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Evolving Graphs by Singular Weighted Curvature

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Dedicated to Professor Rentaro Agemi on the Occasion of His 60th Birthday

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Contents

Abstract

A new notion of solutions is introduced to study degenerate nonlinear parabolic equations in one space dimension whose diffusion effect is so strong at particular slopes of the unknowns that the equation is no longer a partial differential equation. By extending the theory of viscosity solutions, a comparison principle is established. For periodic continuous initial data a unique global continuous solution (periodic in space) is constructed. The theory applies to motion of interfacial curves by crystalline energy or more generally by anisotropic interfacial energy with corners when the curves are the graphs of functions. Even if the driving force term (homogeneous in space) exists, the initial-value problem is solvable for general nonadmissible continuous (periodic) initial data.

1. Introduction

We are concerned with degenerate nonlinear parabolic equations (in one space dimension) whose diffusion effect is very strong at particular slopes of unknown functions. A typical example is a quasilinear equation

$$u_t - a(u_x)(W'(u_x))_x = 0, \tag{1.1}$$

where W is a given *convex* function on $\boldsymbol{R}$ but may not be of class C^1 so that its derivative W' may have jumps although W' is nondecreasing; here a is a given nonnegative continuous function and u_t and u_x denote the time and space derivatives of u. We also consider the more general form

$$u_t - a(u_x)((W'(u_x))_x - C(t)) = 0 \tag{1.2}$$

with a given function C, or even the fully nonlinear equation

$$u_t + F(t, u_x, (W'(u_x))_x) = 0 \tag{1.3}$$

with a given function F satisfying monotonicity or degenerate ellipticity condition

$$F(t, p, X) \leqq F(t, p, Y) \quad \text{for } X \geqq Y \tag{1.4}$$

so that (1.3) is degenerate parabolic. At the first glance the evolution law given by these equations is unclear. Since W' may have jumps, so that W'' contains a sum of delta-type functions, the diffusion coefficient $a(u_x)W''(u_x)$ is no longer a function of u_x. For example, if $W(p) = |p|$, then $W''(p)$ is twice the delta function δ. In this case, (1.1) becomes

$$u_t - 2a(u_x)\delta(u_x)u_{xx} = 0,$$

which is, of course, not a classical partial differential equation. So far, this type of equation was analyzed only for a very restrictive class of piecewise linear unknown functions with piecewise linear W [T1, AG1] or only for (1.1) [FG]. Our eventual goal is a synthetic approach to analyze (1.3).

The purpose of this paper is threefold: (i) to introduce a new notion of solutions to (1.3) (where both solutions and W need not be piecewise linear), (ii) to establish a comparison principle for our solutions, and (iii) to prove the unique existence of global-in-time solutions for (1.3) (with (1.4)) when initial data are only continuous and periodic under a very weak regularity assumption on F. For this purpose we extend the theory of viscosity solutions [CIL] to our setting although our equations are not partial differential equations. It turns out that our extended version is suitable for studying (1.3) when W' has jumps.

Our equations (1.1)–(1.3) stem from material sciences and physics as a geometric evolution law of interfacial curves bounding two phases of materials [Gu, Ch]. However, in the present work no knowledge of material science is assumed.

1.1. Notions of Solutions

J. TAYLOR [T1] proposed an evolution law for a special class of piecewise linear closed curves called admissible, which are moved by *crystalline* energy. If the curve is represented as the graph of a function u, the governing equations in [T1] formally correspond to (1.1) with (positive) *piecewise linear* W, where a is assumed to be proportional to W. Independently, ANGENENT & GURTIN [AG1] derived the same evolution laws (for curves) corresponding to (1.2) by establishing a continuum thermomechanical theory of crystal growth with no relation of a and W (even if W is not piecewise linear). However, this class of curves is still restricted. We first reproduce their equation in our setting. Assume, for simplicity, that $W'(p)$ has a jump at $p = 0$. Suppose that $u(t, \cdot)$ is constant on some closed interval $I(t) = (\alpha(t),\ \beta(t))$. Suppose in a neighborhood of $I(t)$ that the x-derivative of u has a definite sign to the left (and right) of $I(t)$ and that it is small so that $u_x(t, x)$ lies outside other jumps of W'. Integrating (1.1) in a neighborhood $(\alpha(t) - \delta,\ \beta(t) + \delta),\ \delta > 0$ of the interval $I(t)$ we get

$$\begin{aligned}\int_{\alpha-\delta}^{\beta+\delta} u_t\,dx &= \int_{\alpha-\delta}^{\beta+\delta} a(u_x)(W'(u_x))_x\,dx\\ &\sim a(0)\int_{\alpha-\delta}^{\beta+\delta}(W'(u_x))_x\,dx \quad (\delta \text{ is small})\\ &= a(0)\{W'(u_x(\beta+\delta)) - W'(u_x(\alpha-\delta))\}.\end{aligned}$$

We postulate that $u_t(t,\ x)$ is *independent* of x on $I(t)$. Sending δ to zero, we now obtain

$$u_t - a(u_x)\Lambda_W(u) = 0 \quad \text{for } x \in I(t) \tag{1.5}$$

with

$$\Lambda_W(u) = \chi\Delta/L,$$
$$\Delta = W'(+0) - W'(-0), \quad L = \beta(t) - \alpha(t),$$
$$W'(+0) = \lim_{\varepsilon\downarrow 0} W'(\varepsilon), \quad W'(-0) = \lim_{\varepsilon\downarrow 0} W'(-\varepsilon);$$

here χ is *the transition number* defined by

$$\chi = \begin{cases} 1 & \text{if both } u_x(\alpha-\delta) \text{ and } u_x(\beta+\delta)\\ & \quad\text{are positive (for small } \delta),\\ -1 & \text{if both are negative,}\\ 0 & \text{otherwise.}\end{cases}$$

The quantity $\Lambda_W(u)$ is called *the weighted curvature* on $I(t)$. Assume for a moment that W is piecewise linear and that the jumps of W' are $p_1 < p_2 < \cdots < p_m$. Assume that u is *an admissible evolving crystal* in a time interval J, i.e.,

(a) $u(t, \cdot)$ is piecewise linear whose slope consists of p_i's;
(b) for each $t \in J$, if $u(t, \cdot)$ has slope p_i on an interval, then the slope of $u(t, \cdot)$ in an adjacent neighborhood is either p_{i+1} or p_{i-1};

(c) u is continuous in space-time and (the abscissa of) a jump of $u_x(t,\cdot)$ moves smoothly in time $t \in J$; jumps do not collide with each other.

We have abused the use of word 'crystal' to represent functions rather than curves. Using an argument similar to that used in deriving (1.5), we find that (1.5) holds on each space (maximal) interval $I_j(t) = (\alpha_j(t),\ \beta_j(t))$, where $u_x(t,\cdot)$ is constant p_i and

$$\Delta = W'(p_i + 0) - W'(p_i - 0), \quad L = \text{the length of } I_j(t);$$

the transition number $\chi = 1$ if both $u_x(\alpha_j - \delta)$ and $u_x(\beta_j + \delta)$ are greater than p_i (for small δ); $\chi = -1$ if both are smaller than p_i; $\chi = 0$ otherwise. This (1.5) is the evolution law we seek (for an admissible evolving crystal) corresponding to (1.1). It turns out that equation (1.5) on each $I_j(t)$ yields a system of ordinary differential equations for the end points of $I_j(t)$ or the length of $I_j(t)$ (cf. [T1, T3, AG1, GirK1, GMHG2]). If the set $\{I_j\}$ is finite or if $u(t,\cdot)$ is periodic in x, then the system of ordinary differential equations has only finitely many unknowns and is solvable locally in time. So, in particular, if initial data satisfy (a) and (b) and are periodic in x, then there is an admissible evolving crystal satisfying (1.5) (locally in time) with these initial data. Some I_j's may disappear at the maximal time t_0, where the ordinary differential equation system is solvable on $(0, t_0)$. Fortunately, if a is positive, $u(t_0,\cdot)$ fulfills (a) and (b) (at t_0) so one can again solve the ordinary differential equation system with initial data $u(t_0,\cdot)$; repeating this argument we extend the solution globally in time [T3, GirK1]. In the terminology of [GMHG2] *there is a global weakly admissible evolving crystal satisfying* (1.5) *with given periodic initial data satisfying* (a) *and* (b) *with* $t = 0$ *provided that* $a > 0$. The same argument applies to (1.2) with a trivial modification [GMHG2].

This approach is good especially for computational purpose. However, there arise at least two fundamental questions:

(I) If initial data do not satisfy (a), (b), i.e., if they are not admissible, what is a natural formulation of solutions to (1.1) or (1.2)?

(II) Is it possible to solve the initial-value problem for (1.1) or (1.2) when W' has jumps but W is not necessarily piecewise linear?

For (1.1) FUKUI & GIGA [FG] introduced a new notion of solutions for general W and general Lipschitz initial data by adapting the theory of nonlinear semigroups initiated by KŌMURA [Ko]. For periodic initial data they constructed a unique global-in-time solution to (1.1). The problems (I), (II) are settled in this case. They rewrote (1.1) in a divergence form

$$u_t - \tilde{W}(u_x)_x = 0$$

by setting

$$\tilde{W}(p) = \int_0^p a(q) W''(q)\, dq$$

and applied the theory of subdifferential equations in periodic L^2 spaces to get solutions. We should ask whether or not a solution of [T1] and [AG1] is also a solution in [FG] when W is piecewise linear. Fortunately, both solutions are consistent. Indeed, in [FG] it is shown that the time derivative of their solutions actually agrees with the one given by (1.5) if u is piecewise linear.

Recently, Elliot, Gardiner & Shätzle [EGS] proved that a weakly admissible evolving crystal satisfying (1.5) on each linear portion of the graph of $u(t,\cdot)$ (called *facet*) is a solution in the sense of [FG] when W is piecewise linear. The solution in [FG] is given by the limit of solutions of (1.1) with W replaced by a regularized W^{ε} approximating W. Thus the evolution law (1.5) for admissible evolving crystals is justified. The behavior of solutions of [FG] is studied both analytically and numerically in [EGS]. Although the theory in [FG] works well for (1.1), it seems very difficult to apply the theory to (1.2) in the presence of C even if C is a (nonzero) constant.

There is another justification of the evolution law (1.5) (for closed curves) by Almgren & Taylor [AT] when W is piecewise linear. They investigated a semi-discretized implicit scheme introduced by [ATW]. The time is discretized and at each time step the value of a solution is given by solving a (non-discretized) variational problem. Their scheme does not require that the solution be piecewise linear at each time step. Their approximate solutions converge as the time grid tends to zero (by taking a subsequence) for general interfacial energy [ATW]. The limit weak solution is called a flat curvature flow. In [AT] it is shown that a weakly admissible evolving crystal (of closed curves) is the unique flat curvature flow with the same initial data provided that two adjacent facets do not vanish simultaneously. This justifies (1.5). For smooth W see also [FK] and [LS]. In particular, it is shown in [FK] that the limit is contained in the level-set flow of [CGG] and [ES].

In this paper we introduce a new notion of solutions for (1.2) or its general form (1.3) with (1.4). Since our theory is interpreted as an extension of the theory of viscosity solutions [CIL], we should define sub- and supersolutions of (1.3) for nonsmooth functions. At issue is the class of test functions we choose so that we define weighted curvatures. We shall always assume that the set of jump discontinuities of W' is a discrete set P. For technical reasons we also assume that $W \in C^2(\boldsymbol{R}\setminus P)$ has bounded second derivatives on each bounded set in $\boldsymbol{R}\setminus P$.

We introduce the notion of (P-) faceted functions in an open interval. Roughly speaking, a piecewise C^1 function $f \in C(\Omega)$ is P-faceted if for each $p \in P$ the set $\{x;\ f'(x) = p\}$ consists of a union of closed (nontrivial) intervals (called *faceted regions*) and the transition number χ is well-defined on each faceted region. We then introduce the class of *P-faceted C^2 functions* on an open interval Ω so that the weighted curvature Λ_W is defined everywhere (§2). Let us give a definition of subsolution at least for continuous functions. We say a continuous function $u : [0,T) \times \overline{\Omega} \to \boldsymbol{R}$ is a *subsolution* of (1.3) in $Q = (0,T) \times \Omega$ if

$$g'(\hat{t}) + F(\hat{t},\, f'(\hat{x}),\, \Lambda_W(f)(\hat{x})) \leqq 0$$

whenever $g \in C^1(0,T),\ f \in C^2_P(\Omega)$ satisfies

$$\max_Q(u-\psi) = (u-\psi)(\hat{t},\,\hat{x}), \quad (\hat{t},\,\hat{x}) \in Q$$

with $\psi(t,x) = g(t) + f(x)$ (called a test function at $(\hat{t},\hat{x})$), where $C^2_P(\Omega)$ denotes the set of all P-faceted C^2 functions on Ω. If $f'(\hat{x}) \in P$, as already explained we set

$$\Lambda_W(f)(\hat{x}) = \chi\Delta/L,$$

where L is the length of faceted region containing $\hat{x}$. If $f'(\hat{x}) \notin P$, then we set

$$\Lambda_W(f)(\hat{x}) = W''(f'(\hat{x}))f''(\hat{x}).$$

It is standard to extend this definition to semicontinuous functions as in §2. A supersolution is defined in the similar way. By *a generalized solution* we mean a sub- and supersolution. It turns out that a generalized solution is consistent with an admissible solution in [T1] and [AG1] when W is piecewise linear. Indeed, it is shown in [GMHG2] that *a weakly admissible evolving crystal satisfying* (1.5) *in each faceted region is a generalized solution in this sense.* The argument can be extended to (1.2) with extra assumptions on a. When C in (1.2) does depend on x, our present theory does not apply. Moreover, the assumption that u_t is constant on $I(t)$ in deriving (1.5) seems to be unnatural [R, GMHG1, GMHG3, GMHG5].

1.2. Comparison Results

It is always crucial to establish a comparison principle in the theory of viscosity solutions. It is, modulo suitable assumptions, of the form:

Comparison Principle. *If u and v are a sub- and supersolution of* (1.3) (*when* (1.4) *holds*) *in $Q = (0,T) \times \Omega$, then $u \leqq v$ in Q provided that $u \leqq v$ on the parabolic boundary $\partial_p Q$ of Q.*

We establish this comparison principle for a bounded open interval Ω under uniform continuity assumptions on F in

$$[0,T'] \times [-K,K] \times \boldsymbol{R} \quad \text{for each } T' < T,\ K > 0.$$

(If F is independent of t, we only need continuity of F.) This result applies to (1.1) and (1.2) when $a \geqq 0$ and C are continuous in $\boldsymbol{R}$ and $[0,T)$, respectively.

The basic strategy of the proof is the same as that for the case when W is smooth. However, several new ideas and extra work are necessary. This is why the proof is long (§§4–7). Since the standard maximum principle for semicontinuous functions [CIL] does not handle our weighted curvature, we establish a maximum principle for faceted functions (§4). To handle semicontinuous functions we need to regularize them by sup-convolutions [CIL]. Unfortunately, the usual sup-convolutions are not good for our purpose. We introduce a sup-convolution by faceted functions and study its properties in §5. A new aspect of our sup-convolution is that if f assumes a local maximum at $\hat{x}$, then its sup-convolution is faceted near $\hat{x}$ (Theorem 5.3). Usually, there is an equivalent definition of subsolutions using semijets. We have to introduce similar equivalent definitions of subsolutions. It is very sensitive how to define time semijets in a neighborhood of a faceted region so that both definitions of subsolutions are equivalent (§6).

With these preparations we prove the Comparison Principle in §7. Let us briefly explain the idea of our proof. We argue by contradiction. Using an extension of the method developed in [CGG] and [Go] we reduce our problem so that our sup-convolutions of both u and $-v$ are faceted at a point we are interested in. We construct suitable test functions of both u and v from these sup-convolutions with the help of equivalent definitions of solutions. Applying the maximum principle for faceted functions and (1.4), we get a contradiction. Note that our method does apply when P is an infinite set. However, it seems difficult to extend our proof to unbounded Ω because so far the method in [CGG] and [Go] requires the boundedness of Ω. If both functions u and v are periodic in space with the same period (independent of time) we also have a comparison principle as shown in §7.

Our comparison principle is totally new even for (1.1) when W' has jumps. In [FG] a comparison principle is proved for their solutions for (1.1). It is obtained as a limit of approximate solutions satisfying a comparison principle. Besides the difference of definitions of solutions, their results are weaker than ours since they assume that both u and v are *solutions* periodic in space. In [GGu] a maximum principle and a comparison principle are proved for admissible evolving crystals. Although they handle curves of finitely many facets with no end points for a comparison principle, their maximum principle applies to yield a comparison principle for weakly admissible evolving crystals satisfying

$$u_t - a(u_x)(\Lambda_W(u) - C(t)) = 0$$

on each faceted region when W is piecewise linear; this equation corresponds to (1.2), of course. (In [GGu] C is assumed to be a constant but the method and result apply to nonconstant C with trivial modifications.) Our results are considered as a natural extension of this type of results since a weakly admissible evolving crystal satisfying (1.5) is a generalized solution [GMHG2].

If the singularities of $W \in C^2(\boldsymbol{R}\backslash P)$ are weak, for example, if W' is Lipschitz continuous, then the comparison principle for viscosity solutions has already been proved in [G]; see also [OhS] and [GSS] for evolutions of closed curves.

1.3. Existence Results

In our formulation we shall prove the (unique) existence of a global-in-time, continuous, periodic-in-space solution of (1.3) if initial data are continuous and periodic. This answers our questions (I) and (II) at least for periodic data. For the homogeneous Dirichlet problem on a bounded interval Ω of (1.3) we obtain some global existence results by reducing the problem to a periodic situation. This result applies to (1.1). However, it does not apply to (1.2) even if C is a (nonzero) constant. This is not a technical restriction. In fact, even for smooth W with $W'' > 0$, a local solution may break down at the boundary $\partial\Omega$ and boundary detachment phenomena may occur (cf., e.g., [KK]). On the contrary, our global solution for the Dirichlet problem actually attains zero at $\partial\Omega$ for all time. This is an intrinsic reason why our global result does not apply to (1.2).

We do not pursue the boundary-value problems in this paper except for a few remarks.

To show existence results we establish the Perron-type existence theorem to our equations. This is by now standard in the theory of viscosity solutions [CIL] for partial differential equations. However, in our setting, we must prove it when the test function is faceted at a point of interest. Unfortunately, this is not a trivial modification of the standard results. We must modify the test function ψ in a class of C^2 P-faceted functions in space so that $\max_Q(u-\psi)$ is attained only at one time $\hat{t}$ and at one faceted region. We introduce canonical modification of C^2 P-faceted functions for this purpose. Another property we use in the proof is

$$W''(f'(x_n))f''(x_n) \to 0 \quad \text{as } x_n \to x_0$$

for $f \in C_P^2(\Omega)$, where x_0 is the boundary of a faceted region of f and $f'(x_n) \notin P$. This convergence follows from the fact that W'' is bounded in every bounded set in $\boldsymbol{R}\backslash P$ and that $f \in C_P^2(\Omega)$. We shall prove the Perron-type existence results in §8. In the last section we construct a sub- and supersolution for given initial data. This together with the Perron-type existence results and comparison results yield an existence result for periodic data. To construct sub- and supersolutions we modify the method developed in [CGG] and improved by [IS].

Our existence result for general continuous (not necessarily Lipschitz continuous) initial data (periodic in x) is new even for (1.1). The existence result in [FG] needs Lipschitz continuity of initial data. We do not know whether a solution in [FG] is continuous in space-time although it applies to all convex W. Our solution is consistent with a solution in [FG] as we proved in [GMHG3].

When W is piecewise linear, a solution with nonadmissible data (question II) for (1.1) has been studied in [T3] and [EGS]. If initial data are piecewise linear nonadmissible, a 'solution' is constructed in [T3] by solving ordinary differential equations. It turns out [EGS] that it is a solution in the sense of [FG] for (1.1). Even for some piecewise C^1 initial data, a solution given by ordinary differential equations is proposed in [T3, EGS]. However, the initial data are very restricted. Instead of proving that each proposed solution is a solution in our sense, we give an example of solutions with nonadmissible data in §2.

1.4. Background of Problems

Surface-energy-driven motion of interfaces has attracted many mathematicians and physicists to study the evolution of phase boundaries Γ_t such as a surface of crystal. Let $\boldsymbol{n}$ denote the unit normal vector field determining the orientation of Γ_t. We assume that Γ_t is a curve in the plane. Let V denote the normal velocity in the direction of $\boldsymbol{n}$. If V depends on local geometry, a typical evolution equation is of the form

$$V = -\frac{1}{\beta(\boldsymbol{n})}\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\partial_i\gamma(\boldsymbol{n})) + C(t)\right) \quad \text{on } \Gamma_t. \tag{1.6}$$

Here $\gamma : \boldsymbol{R}^2 \to \boldsymbol{R}$ is of the form

$$\gamma(q) = |q|\gamma_0(q/|q|), \quad q = (q_1, q_2) \in \boldsymbol{R}^2, \ q \neq 0,$$

and γ_0, β are positive functions defined on the unit circle. In (1.6) $\partial_i \gamma$ denotes the partial derivative $\partial\gamma/\partial q_i$ as a function on $\boldsymbol{R}^2$. The quantity γ_0 is called *the interfacial energy density*, while β is called *the kinetic coefficient*. The function $C(t)$ is given. Physically it describes the bulk free energy of crystal relative to that of the other phase. Typically, this is the difference of temperature between two phases on Γ_t or the difference of pressure. The equation (1.6) was derived mathematically by ANGENENT & GURTIN [AG1] (see also [Gu]) from axioms of continuum thermomechanical theory. In the physics literature, (1.6) was first formulated by MÜLLER-KRUMBHAAR et al. [MBK] as a gradient flow of free energy in the spirit of the time-dependent Ginzburg-Landau theory. This model is good if the crystal surrounded by Γ_t is so small that surface effects dominate bulk effects, since in this case we may assume that bulk energy $C(t)$ is given and is independent of the space variable. If γ_0 is a positive constant, γ_0 is called *isotropic*. If both γ_0 and β are isotropic, (1.6) with $C = 0$ becomes the famous curve-shortening equation. We focus on the initial-value problem for anisotropic curve shortening equation (1.6) when the interfacial energy may have singularities. To classify the problems it is convenient to recall *the Frank diagram* of γ_0:

$$\mathscr{F} = \{(q_1, q_2) \in \boldsymbol{R}^2; \ \gamma(q) = 1\}.$$

We always assume that $\mathscr{F}$ is convex, so that (1.6) is degenerate parabolic at least formally. There are three typical situations as explained in [GMHG1].

(1) $\mathscr{F}$ is smooth and has positive curvature; γ_0 is called *a strict convex smooth energy*.
(2) $\mathscr{F}$ is at most of class $C^{1,1}$ and of class C^2 except at finitely many points. The curvature of $\mathscr{F}$ is bounded but may be zero somewhere; γ_0 is called *a singular energy without corners*.
(3) $\mathscr{F}$ is of class C^2 except at finitely many points but is not of class C^1. The curvature of $\mathscr{F}$ is bounded but may be zero somewhere; γ_0 is called *an energy with corners*.

An isotropic energy is a typical example of a strictly convex smooth energy. In the case of (1) the equation (1.6) is quasilinear (nondegenerate) parabolic so the classical theory [LSU] applies to get a local-in-time smooth solution to (1.6) provided that β is smooth. Even if C exists, it is possible to extend the solution globally in time in the level-set sense ([CGG], [ES]). Note that the level-set method applies to (1.6) if γ_0 is of class C^2 and is convex; the strict convexity is unnecessary [CGG, GGo]. There is by now an extensive literature for the case (1). We suggest that the reader consult the book [Gu], the review [TCH], the review [GMHG1] and the references therein.

Even in the case of (2) the level-set method yields global generalized solutions for given initial curves [OhS, GSS]. A local existence of "strong" solutions is established by [AG2]. If Γ_t is given as a graph of $y = u(t, x)$, then (1.6) becomes

$$u_t - a(u_x)((W'(u_x))_x - C(t)) = 0 \tag{1.7}$$

with

$$
\begin{aligned}
a(p) &= (1+p^2)^{1/2} M(p),\\
1/M(p) &= \beta(-p/(1+p^2)^{1/2}, (1+p^2)^{-1/2}),\\
W(p) &= \gamma(-p, 1).
\end{aligned} \tag{1.8}
$$

Here (2) says that W is in $C^{1,1}$ and is in C^2 except at finitely many points. In [G] properties of a global generalized solution are studied as well as comparison and existence results. See [GMHG1] and the references therein.

In the case of (3) the existing level-set method does not apply to evolving closed curves so far[1]. We discuss the case when Γ_t is represented as the graph of functions. Our W in (1.8) satisfies our assumption of regularity if (and only if) γ_0 is an energy with corners. This paper extends the theory of viscosity solutions to establish fundamental comparison and existence of solutions so that it applies to this setting.

A typical example of an energy with corners is *a crystalline energy*, where $\mathscr{F}$ is assumed to be a convex polygon. For (1.7) this is the case when W is piecewise linear with finite jumps of W'. As mentioned before, its evolution law (1.6) is reduced to a system of ordinary differential equations for admissible evolving crystals. This was first observed by TAYLOR [T1] (for $\beta \cdot \gamma = \text{const.}$) and independently by ANGENENT & GURTIN [AG1]. Their evolution is qualitatively similar to the case when γ_0 is smooth. For example, we have a comparison principle for admissible evolving crystals 'solving' (1.6) [GGu]. If $\beta \cdot \gamma = \text{const.}$ and $C \equiv 0$, a closed convex admissible evolving crystal shrinks to a point and the way of shrinking is asymptotically similar to the Wulff shape of γ provided that the initial polygon has more than five corners [St] as conjectured by TAYLOR [T1]. The Stefan-type problem with crystalline interfacial energy is studied in [Ry]. Although it is interesting to study the behavior of our solutions, we do not discuss them in this paper. Recently, *surface* evolutions by crystalline energy were analyzed by [GGuM]. Among other results, a comparison principle for admissible evolution was established there. For the background of a crystalline energy see the review of TAYLOR [T2].

It is a geometrically natural idea to approximate a strictly convex smooth energy by a crystalline one. In fact, this approximation has been used in the calculation of curve evolutions. If Γ_t is a graph, the convergence result of [FG] applies to (1.1). It says that an approximate solution actually converges to the true solution with no convergence rate. At the same time GIRÃO & KOHN [GirK1] studied this problem and obtained the convergence rate in the Sobolev space H^1 (for Dirichlet and Neumann problems). For convex closed curves GIRÃO [Gir] obtained a convergence rate in the topology of Hausdorff distance. Recently this result was extended by USHIJIMA & YAZAKI [UY] for the equation $V = -(\operatorname{div} \boldsymbol{n})^\alpha$ on Γ_t with $\alpha > 0$. See also the review paper [GirK2]. Unfortunately, these works do not apply if C exists in (1.2). We shall discuss convergence result

[1] Recently, based on the results in the present paper, it turns out that the level-set method can be extended in the case of (3). We shall discuss this topic in one of our forthcoming papers.

even for general equations (1.3) in the realm of viscosity solutions in [GMHG4, GMHG5].

In the crystal growth problem there often arises an energy γ_0 with corners so that its Wulff shape has a flat portion called a facet. It is natural to consider such a γ_0 if the temperature is lower than the roughening temperature. Although this phenomenon is common in all crystal growth [Ch], a typical example is the growth of crystals of Helium. If facets exists in evolving crystals, it is explained in physics [Ch] that the velocity on facets is not proportional to the chemical potential difference $\Delta\mu$ which is the weighted curvature plus the pressure difference, although the velocity is nondecreasing in $\Delta\mu$ and it is zero for $\Delta\mu = 0$. If $\Delta\mu$ is small, the velocity may be zero in some situation. This is a reason why we consider

$$u_t + F(u_x,\ \Lambda_W(u) + C) = 0$$

or its more general form (1.3) instead of (1.2).

2. Definition of Generalized Solutions

This section establishes conventions of notation and introduces several notions of functions and weighted curvature. The goal of this section is to define generalized solutions for evolution equations with singular interfacial energy.

2.1. Assumption (Set P and function W). Let P be a closed discrete set in $\boldsymbol{R}$. In other words P is either a finite set or a countable set having no accumulation points in $\boldsymbol{R}$. If P is nonempty, P is of form $\{p_j\}_{j=1}^m$, $\{p_j\}_{j=-\infty}^{\infty}$, $\{p_j\}_{j=-\infty}^{-1}$, or $\{p_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} p_j = \infty$, $\lim_{j\to-\infty} p_j = -\infty$, where the p_j's are indexed in increasing order $p_j < p_{j+1}$ and m is a positive integer. Let W be a convex function on $\boldsymbol{R}$ with values in $\boldsymbol{R}$. We assume that W is of class C^2 outside P. Moreover, we assume that W'' is bounded in any compact set except all points in P.

These assumptions on P and W hold throughout this paper.

2.2. Definition. Let Ω be an open interval. A function f in $C(\Omega)$ is called *faceted* at x_0 with *slope* p in Ω if there is a closed nontrivial finite interval $I(\subset \Omega)$ containing x_0 such that f agrees with an affine function

$$l_p(x) = p(x - x_0) + f(x_0)$$

in I and $f(x) \not\equiv l_p(x)$ for all $x \in J \setminus I$ with some neighborhood $J(\subset \Omega)$ of I. The interval I is called a *faceted region* of f containing x_0 and is denoted by $R(f, x_0)$. A function f is called P*-faceted* at x_0 in Ω if f is faceted at x_0 in Ω with some slope p belonging to P.

2.3. Definition. Let x_0 be a point in Ω. For $f \in C(\Omega)$ we set

$$\Lambda_W(f, x_0) = W''(f'(x_0))\, f''(x_0)$$

if f is twice differentiable at x_0 and $f'(x_0) \notin P$, and set

$$\Lambda_W(f, x_0) = \frac{\chi}{L}\Delta_i$$

if f is P-faceted at x_0 in Ω with slope p_i, where $\Delta_i = W'(p_i + 0) - W'(p_i - 0)$. We call the value $\Lambda_W(f, x_0)$ the *weighted curvature of* f *at* x_0. This value is invariant under the addition of affine functions to W. Here $L = L(f, x_0)$ is the length of the faceted region I containing x_0 and $\chi = \chi(f, x_0)$ is the *transition number* defined by

$$\chi = \begin{cases} +1 & \text{if } f \geqq l_{p_i} \text{ in } J, \\ -1 & \text{if } f \leqq l_{p_i} \text{ in } J, \\ 0 & \text{otherwise} \end{cases}$$

for some neighborhood J of the facet region I. We often write $\Lambda_W(f, x_0)$ as $\Lambda_W(f)(x_0)$ to emphasize that this quantity is a function of x_0.

For later convenience, we introduce *the left transition number* $\chi_- = \chi_-(f, x_0)$, and *the right transition number* $\chi_+ = \chi_+(f, x_0)$ by

$$\chi_- = \begin{cases} +1 & \text{if } f \geqq l_{p_i} \text{ in } \{x \in J; x \leqq x_0\}, \\ -1 & \text{if } f \leqq l_{p_i} \text{ in } \{x \in J; x \leqq x_0\}, \end{cases}$$

$$\chi_+ = \begin{cases} +1 & \text{if } f \geqq l_{p_i} \text{ in } \{x \in J; x \geqq x_0\}, \\ -1 & \text{if } f \leqq l_{p_i} \text{ in } \{x \in J; x \geqq x_0\}. \end{cases}$$

By definition, $\chi = \frac{1}{2}(\chi_+ + \chi_-)$.

2.4. Definition. A function $f \in C^2(\Omega)$ belongs to class $C_P^2(\Omega)$ if f is P-faceted at x_0 in Ω whenever $f'(x_0)$ belongs to P. Let T be a positive number. For $Q = (0, T) \times \Omega$ with $T > 0$ let $A_P(Q)$ be the set of functions on Q of the form

$$f(x) + g(t), \quad f \in C_P^2(\Omega), \quad g \in C^1(0, T).$$

An element of $A_P(Q)$ is called *an admissible function*.

2.5. Assumptions (Function F)**.** Let F be a function from $[0, T) \times \boldsymbol{R} \times \boldsymbol{R}$ to $\boldsymbol{R}$. We often assume

(F1) F is continuous in $[0, T) \times \boldsymbol{R} \times \boldsymbol{R}$ with values in $\boldsymbol{R}$,
(F2) $F(t, p, X) \leqq F(t, p, Y)$ for $X \geqq Y, t \in [0, T), p \in \boldsymbol{R}$ (degenerate ellipticity),
(F3) For each $K > 0$ and $T' < T$, F is uniformly continuous in $[0, T'] \times [-K, K] \times \boldsymbol{R}$.

We explicitly state these assumptions when needed.

We can now define our generalized solution in the viscosity sense.

2.6. Definition. A real-valued function u on Q is a (*viscosity*) *subsolution* of

(E) $$u_t + F(t, u_x, \Lambda_W(u)) = 0 \quad \text{in } Q$$

if the upper-semicontinuous envelope $u^* < \infty$ in $[0, T) \times \overline{\Omega}$ and

$$\psi_t(\hat{t},\hat{x}) + F(t,\psi_x(\hat{t},\hat{x}),\Lambda_W(\psi(\hat{t},\cdot),\hat{x})) \leqq 0 \tag{2.1}$$

whenever $(\psi,(\hat{t},\hat{x})) \in A_P(Q)\times Q$ fulfills

$$\max_Q(u^* - \psi) = (u^* - \psi)(\hat{t},\hat{x}). \tag{2.2}$$

Here

$$u^*(t,x) = \lim_{\varepsilon\downarrow 0}\sup\{u(s,y); |s-t| < \varepsilon, |x-y| < \varepsilon, (s,y)\in Q\}$$

for $(t,x) \in \overline{Q}$ and $u_* = -(-u)^*$. A (*viscosity*) *supersolution* is defined by replacing $u^*(<\infty)$ by the lower-semicontinuous envelope $u_*(> -\infty)$, max by min and the inequality in (2.1) by the opposite one. If u is both a sub- and supersolution, u is called a *viscosity solution* or a *generalized solution*. Hereafter we avoid using the word viscosity. Function ψ satisfying (2.2) is called *a test function* of u at $(\hat{t},\hat{x})$.

The following propositions are easily derived.

2.7. Proposition (Addition by Affine Functions). *Let Ω be an open interval. Let u be a sub- or supersolution of* (E)*. Then $v(t,x) = u(t,x) - l(x)$ is respectively a sub- or supersolution of*

$$v_t + F(t, v_x + A, \Lambda_{W_A}(v)) = 0,$$

where

$$l(x) = Ax + B \quad \textit{for some real number } A \textit{ and } B,$$
$$W_A(q) = W(q+A) \quad \textit{for all } q \in \boldsymbol{R},$$
$$P_A = \{p - A;\ p \in P\}.$$

2.8. Proposition. *If $\psi \in A_P(Q)$ satisfies* (2.1) *at each point $(\hat{t},\hat{x}) \in Q$, then ψ is a subsolution of* (E) *in Q provided that* (F2) *holds.*

2.9. Example of Equations. Let a function F be of the form

$$F(t,p,X) = -a(p)(X + C(t,p))$$

with a continuous nonnegative function a and a continuous function C in $[0,T)\times \boldsymbol{R}$. Then conditions (F1)–(F3) are fulfilled. Equation (E) becomes

$$u_t - a(u_x)(\Lambda_W(u) + C(t,u_x)) = 0. \tag{2.3}$$

The term $\Lambda_W(u) + C$ is called *the weighted curvature with driving force* and is denoted by $\Lambda_W(u;C)$ as in [GMHG1]. A more general form of F is

$$F(t,p,X) = G(t,p,X + C(t,p)),$$

where $G : [0,T)\times \boldsymbol{R}\times \boldsymbol{R} \to \boldsymbol{R}$. It is easy to see that F fulfills (F1) or (F3) provided that G respectively satisfies (F1) or (F3) where C is continuous in $[0,T)\times \boldsymbol{R}$. Clearly if G satisfies (F2), so does F. Equation (E) now becomes

$$u_t + G(t,u_x,\Lambda_W(u) + C(t,u_x)) = 0.$$

Such an equation is important for describing evolutions of crystals of Helium below the roughening temperature (where C is independent of p). As pointed out in [R] (see also [GMHG3]) when C depends on the spatial variable x, it is not natural to define $\Lambda_W(u, C)$ just as a sum $\Lambda_W(u) + C$.

2.10. Remark. Equation (2.3) has many representations for different W's. For example, if

$$a(p)W''(p) = a_0(p)W_0''(p) \tag{2.4}$$

in the distributional sense with some nonnegative continuous function a_0 and convex function W_0, then (2.3) can be formally written as

$$u_t - a_0(u_x)\Lambda_{W_0}(u) - a(u_x)C(t, u_x) = 0, \tag{2.5}$$

which is also an example of (E). Fortunately, it is not difficult to see that u is a subsolution of (2.3) if and only if u is a subsolution of (2.5). So our definition of solutions is independent of the representation of the equation. By the way, there are many choices of a_0 and W_0 satisfying (2.4). Indeed, for a given $a_0(p)$, we have W_0 satisfying (2.4) by defining W_0 by

$$W_0(p) = \int^p \int^q a(z)W''(z)a_0(z)^{-1}\, dz\, dq.$$

2.11. Examples of Solutions When W is Piecewise Linear. We next consider several special solutions of (2.3) when $a > 0$ and W is piecewise linear. We say a function v on $\boldsymbol{R}$ is an *admissible crystal* if v is P-faceted at any point of $\boldsymbol{R}$ and the slopes of adjacent faceted regions should be adjacent in P. This definition is the same as in [GMHG2]. Let u be *an admissible evolving crystal on a time interval* I', i.e., $u(t, \cdot)$ is an admissible crystal with $u \in C(I' \times \boldsymbol{R})$, and jumps of u_x moves smoothly in time for $t \in I'$ and do not collide with each other. For an admissible evolving crystal u equation (2.3) has a meaning on each faceted region of $u(t, \cdot)$. This equation agrees with those derived by [T1] and [AG1] in a slightly different setting and by [GirK1] in this setting (actually with $C = 0$ and finite P but these restrictions are inessential). In [GMHG2] we prove that *an admissible evolving crystal u in* $(0, T) \times \boldsymbol{R}$ *satisfying* (2.3) *is a generalized solution of* (2.3) *with* $Q = (0, T) \times \boldsymbol{R}$ if $C = 0$. The same assertion is still valid under some reasonable assumption on a even if C does not vanish identically. In [GMHG2] P was assumed to be finite but the proof works for general P. Even if u is a weakly admissible evolving crystal, i.e., if u is an admissible evolving crystal on $[t_k, t_{k+1}), k = -1, 0 \dots, h$, for some $0 = t_{-1} < t_0 < \cdots < t_{h+1} = T$ and u is continuous across $t = t_k, k = 0 \dots, h$, then u is a generalized solution of (2.3) with $\Omega = \boldsymbol{R}$ provided that u satisfies (2.3) on $(t_k, t_{k+1}), k = -1, 0 \dots, h$ (under some assumptions on a unless $C = 0$). This is also proved in [GMHG2]. In the definition of weakly admissible evolving crystals of [GMHG2], the open interval (t_l, t_{l+1}) should be replaced by $[t_l, t_{l+1})$.

In many cases for a given initial function u_0 which is an admissible crystal, we see there is a unique global-in-time weakly admissible evolving crystal u satisfying (2.3) on $(t_k, t_{k+1}), k = -1, 0 \dots, h$ by solving a system of ordinary differential equations. A typical situation is that u_0 is periodic in x. Since a weak admissible evolving crystal solving (2.3) is our generalized solution of (2.3), there are many nontrivial examples of solutions.

2.12. Examples of Solutions with Nonadmissible Initial Data. When W is piecewise linear, a solution with nonadmissible data for (1.1) has been studied in [T3] and [EGS]. For piecewise linear but nonadmissible initial data a solution is constructed in [T3] by solving ordinary differential equations. However, if initial data are not piecewise linear, creation of new facets also may occur but such an example is not well examined although there is a heuristic explanation in [T3].

We give here a simple example of a solution with such initial data for (1.1) with $W(p) = c|p|$ with $c > 0$:

$$u_t - ca(u_x)(\operatorname{sgn} u_x)_x = 0, \tag{2.6}$$

where $a \geqq 0$ is continuous; we assume that $a(0) > 0$ since otherwise (2.6) becomes $u_t = 0$ everywhere. We consider initial data $u_0 \in C(\boldsymbol{R})$ of the form

$$u_0(x) = \begin{cases} A(x), & x \leqq \alpha_0, \\ h_0, & \alpha_0 \leqq x \leqq \beta_0, \\ B(x), & \beta_0 \leqq x \end{cases} \tag{2.7}$$

with

$$\alpha_0 \leqq \beta_0,$$

$$\begin{aligned} &A \in C^1(-\infty, \alpha_0) \cap C(-\infty, \alpha_0], && A' < 0, \\ &B \in C^1(\beta_0, \infty) \cap C[\beta_0, \infty), && B' > 0, \end{aligned}$$

$$A(\alpha_0) = B(\beta_0) = h_0 \quad \text{so that } u_0 \text{ is continuous.}$$

We set

$$D(k) = \int_{h_0}^{k} \{B^{-1}(\eta) - A^{-1}(\eta)\} d\eta,$$

where $A^{-1}, B^{-1} \in C^1(h_0, \infty) \cap C[h_0, \infty)$ denote the inverse functions of A and B. Note that

$$B^{-1}(\eta) > \beta_0 \geqq \alpha_0 > A^{-1}(\eta), \quad \eta > h_0$$

so that $D'(k) > 0$ for $k > h_0$. The value $D(k)$ is the area enclosed by $y = k$ and $y = u_0(x)$. The inverse function D^{-1} is well defined and $D^{-1} \in C^1(h_0, \infty) \cap C[h_0, \infty)$. We then take $u \in C(\overline{Q})$ ($Q = (0, \infty) \times \boldsymbol{R}$) of the form

$$u(t, x) = \begin{cases} A(x), & x \leqq \alpha(t), \\ h(t), & \alpha(t) \leqq x \leqq \beta(t), \\ B(x), & \beta(t) \leqq x \end{cases} \tag{2.8}$$

with

$$h(t) = D^{-1}(2a(0)ct), \tag{2.9}$$

$$\begin{aligned} &\alpha(t) = A^{-1}(h(t)), \quad \alpha(0) = \alpha_0, \\ &\beta(t) = B^{-1}(h(t)), \quad \beta(0) = \beta_0 \quad \text{with } \alpha_0 \leqq \beta_0. \end{aligned} \tag{2.10}$$

Since $h(t) > h_0$ for $t > 0$ and

$$B^{-1}(\eta) > \beta_0 \geqq \alpha_0 > A^{-1}(\eta), \quad \eta > h_0,$$

we see that $\alpha(t) < \beta(t)$ for $t > 0$. Thus the function u is well defined and $u \in C(\overline{Q})$ since $h \in C[0,\infty)$ so that $\alpha, \beta \in C[0,\infty)$. The interval $[\alpha(t), \beta(t)]$ is a faceted region of $u(t,\cdot), t > 0$ and $\alpha'(t) < 0 < \beta'(t)$ for $t > 0$. To convince the reader that (2.8) satisfies (2.6), we calculate u_t on $(\alpha(t), \beta(t))$. By (2.9), (2.10) and definition of D we see that

$$\beta(t) - \alpha(t) = A^{-1}(h(t)) - B^{-1}(h(t)) = D'(h(t)), \quad t > 0. \tag{2.11}$$

Since

$$u_t(t,x) = h'(t) \quad \text{for } x \in (\alpha(t), \beta(t))$$

this yields

$$\begin{aligned} u_t(t,x) = h'(t) &= (D^{-1})'(2a(0)ct) \cdot 2a(0)c \\ &= \frac{2a(0)c}{D'(h(t))} = \frac{2a(0)c}{\beta(t) - \alpha(t)} = a(0)\Lambda_W(u(t,\cdot),x), \end{aligned}$$

where $W(p) = c|p|$.

Lemma. *The function $u \in C(\overline{Q})$ in* (2.8) *is a generalized solution of* (2.6) *in Q. Its initial function is u_0 in* (2.7). *Moreover $-u$ is a generalized solution of* (2.6) *in Q with initial data $-u_0$.*

This is true even if $\alpha_0 = \beta_0$. In this case a new faceted region $[\alpha(t), \beta(t)]$ is created instantaneously. The proof of the lemma is easy but we give it for completeness.

Proof. Step 1. We show that the function u is a subsolution of (2.6) in Q: Let $\psi(t,x)$ be of the form

$$\psi(t,x) = f(x) + g(t) \quad f \in C_P^2(\boldsymbol{R}), \;\; g \in C^1(0,\infty) \tag{2.12}$$

and suppose that

$$\max_Q(u - \psi) = (u - \psi)(\hat{t}, \hat{x}), \hat{t} > 0;$$

we do not take u^* since u is continuous. If $f'(\hat{x}) = 0$, then f is faceted at $\hat{x}$ and by geometry we see that

$$\Lambda_W(u(\hat{t},\cdot),\hat{x}) \leqq \Lambda_W(f,\hat{x}), \quad g'(\hat{t}) = h'(\hat{t}).$$

By (2.11) we have

$$g'(\hat{t}) - a(0)\Lambda_W(f,\hat{x}) \leqq h'(\hat{t}) - a(0)\Lambda_W(u(\hat{t},\cdot),\hat{x}) = 0.$$

If $f'(\hat{x}) \not\equiv 0$ so that $\hat{x} < \alpha(\hat{t})$ or $\hat{x} > \beta(\hat{t})$, then

$$\Lambda_W(f, \hat{x}) = 0$$

since $W''(p) = 0$ for $p \neq 0$. Clearly $g'(\hat{t}) = u_t(\hat{t}, \hat{x}) = 0$, so we have

$$g'(\hat{t}) - a(0)\Lambda_W(f, \hat{x}) = 0$$

in this case. Since $u \in C(\overline{Q})$, u is now a subsolution of (2.6) in Q.

Step 2. We show that the function u is a supersolution of (2.6) in Q: Let $\psi(t, x)$ be as in (2.12) and suppose that

$$\min_Q (u - \psi) = (u - \psi)(\hat{t}, \hat{x}), \quad \hat{t} > 0. \tag{2.13}$$

(1) If $f'(\hat{x}) = 0$ and $\hat{x} \neq \alpha(\hat{t}), \hat{x} \neq \beta(\hat{t})$, then

$$\Lambda_W(u(\hat{t}, \cdot), \hat{x}) \geqq \Lambda_W(f, \hat{x}), \quad g'(\hat{t}) = h'(\hat{t}),$$

so that by (2.11) we have

$$g'(\hat{t}) - a(0)\Lambda_W(f, \hat{x}) \geqq h'(\hat{t}) - a(0)\Lambda_W(u(\hat{t}, \cdot), \hat{x}) = 0.$$

(2) If $f'(\hat{x}) \neq 0$ and $\hat{x} \neq \alpha(\hat{t}), \hat{x} \neq \beta(\hat{t})$, then

$$\Lambda_W(f, \hat{x}) = 0, \quad g'(\hat{t}) = 0$$

as in Step 1. Thus

$$g'(\hat{t}) - a(0)\Lambda_W(f, \hat{x}) = 0 - 0 = 0.$$

(3) It remains to study the case when either $\hat{x} = \alpha(\hat{t})$ or $\hat{x} = \beta(\hat{t})$. If $f'(\hat{x}) = 0$ and $\operatorname{int} R(f, \hat{x})$ intersects $(\alpha(\hat{t}), \beta(\hat{t}))$, then the situation is reduced to (1), so we may always assume that

$$\Lambda_W(f, \hat{x}) \leqq 0$$

including the case that $f'(\hat{x}) \neq 0$ (which implies that $\Lambda_W(f, \hat{x}) = 0$). We may assume that $\hat{x} = \alpha(\hat{t})$ since the case $\hat{\beta}(\hat{t})$ can be treated in the same way. Since $u(\alpha(\hat{t}), t) = A(\alpha(\hat{t}))$ for $t < \hat{t}$, (2.13) implies that $g'(\hat{t}) \geqq 0$. This now yields

$$g'(\hat{t}) - a(0)\Lambda_W(f, \hat{x}) \geqq 0.$$

Note that $g'(\hat{t})$ may not equal $h'(\hat{t})$ when $f'(\hat{x}) \neq 0$. Since (2.6) is invariant if we replace u by $-u$, it follows that $-u$ also satisfies (2.6). Clearly $u(0, x) = u_0(x)$ so the proof is now complete. □

Remarks. (i) In [EGS] a piecewise C^1 initial function u_0 with finitely many P-faceted regions was considered. Outside faceted regions u_0 was essentially in C^1 and $u_0' \notin P$. The authors gave a local-in-time solution by solving ordinary differential equations and proved that their solution is a solution in the sense of [FG]. We remark that our lemma essentially implies that their solution is our generalized solution. Note that the situation is localized near each facet so the proof of our lemma applies. Our lemma also allows the situation that new facets are created instantaneously.

(ii) Our solution (2.8) is not of class C^1 in space for all $t > 0$ even if the initial function u_0 is smooth. This shows that the solution $u(t, \cdot)$ may lose smoothness instantaneously. When $\alpha_0 = \beta_0 (= 0)$, the growth of the facet depends on the behavior of A and B near $\alpha_0 = 0$. To see this, we for simplicity assume that $B(x) = mx^r, A(x) = B(-x) \quad (m, r > 0)$ and $h_0 = 0$. A direct computation shows that

$$D(k) = \frac{2}{m^r} \int_0^k \eta^{1/r} d\eta = m^{-1/r} d_0 k^{1+1/r}, \quad d_0 = \frac{2r}{1+r}$$

which yields

$$h(t) = m^{1/(1+r)} \{2d_0^{-1} a(0) ct\}^{r/(1+r)},$$

$$\beta(t) = \{2m^{-1} d_0^{-1} a(0) ct\}^{1/(1+r)} \quad (= -\alpha(t)).$$

We conclude this section by giving another example of a solution of the equation

$$u_t - ca(u_x)(\text{sgn } (u_x - p_0))_x = 0 \tag{2.14}$$

for a given number $p_0 \in \boldsymbol{R}$. We assume that $a > 0$ everywhere in this example. We consider (2.14) with initial data

$$u_0(x) = \mu \sin(\nu x), \quad \mu > 0, \ \nu > 0. \tag{2.15}$$

Assume that $0 \leqq p_0 < \mu\nu$. As in the previous example we obtain an explicit form of the solution with initial data. Let α_0 be the unique number that satisfies

$$v_0'(\alpha_0) = 0 \quad \text{with} \ -\pi/\nu < \alpha_0 < 0,$$

where

$$v_0(x) = u_0(x) - p_0 x.$$

Let α_1 be the unique number that satisfies

$$v_0(\alpha_1) = 0 \quad \text{with} \ -\pi/\nu \leqq \alpha_1 < 0,$$

so that $\alpha_1 \leqq \alpha_0$. We set

$$\begin{aligned} D(k) &= \int_{v_0(\alpha_0)}^{k} \{B^{-1}(\eta) - A^{-1}(\eta)\} d\eta \\ &= \int_{A^{-1}(k)}^{B^{-1}(k)} \{k - v_0(x)\} dx, \quad 0 \geqq k \geqq v_0(\alpha_0), \end{aligned}$$

where $A(x) = v_0(x)$ for $x, \alpha_1 \leqq x \leqq \alpha_0$ and $B(x) = v_0(x)$ for $x, \alpha_0 \leqq x \leqq 0$. The value $D(k)$ is the area enclosed by $y = k$ and $y = v_0(x)$ with $\alpha_1 < x < 0$. Let $T > 0$ be the number defined by

$$T = \frac{D(0)}{2a(p_0)c} = \frac{1}{2a(p_0)c} \left\{ \frac{\mu}{\nu}(1 - \cos(\nu\alpha_1)) - \frac{p_0 \alpha_1^2}{2} \right\}.$$

We set

$$h(t) = D^{-1}(2a(p_0)ct) \quad \text{for } 0 \leqq t \leqq T,$$

so that $h(0) = v_0(\alpha_0), h(T) = 0, h'(t) > 0$ for $0 < t < T$. We then set

$$\alpha(t) = A^{-1}(h(t)), \quad 0 \leqq t \leqq T,$$

$$\beta(t) = B^{-1}(h(t)), \quad 0 \leqq t \leqq T,$$

so that

$$\alpha(0) = \alpha_0, \quad \alpha(T) = \alpha_1, \quad \beta(0) = \alpha_0, \quad \beta(T) = 0.$$

As in the proof of the Lemma one can prove that the $2\pi/\nu$-periodic (in space) continuous function u defined by

$$u(t,x) = \begin{cases} u_0(x), & -\pi/\nu < x < \alpha(t), \\ u_0(\alpha(t)) + p_0(x - \alpha(t)), & \alpha(t) \leqq x \leqq \beta(t), \\ u_0(x), & \beta(t) < x < -\beta(t), \\ u_0(-\alpha(t)) + p_0(x + \alpha(t)), & -\beta(t) \leqq x \leqq -\alpha(t), \\ u_0(x), & -\alpha(t) < x \leqq \pi/\nu \end{cases} \quad \text{for } 0 \leqq t \leqq T,$$

$$u(t,x) = u(T,x) \qquad \text{for } t > T$$

is the generalized solution of (2.14) in $Q = (0,\infty) \times \boldsymbol{R}$ with initial data $u_0(x) = \mu \sin(\nu x)$. Note that $u(t,x)$ becomes a stationary solution in a finite time. If $|p_0| \geqq \mu\nu$, the initial function $u_0(x)$ itself is the stationary solutions (If $-\mu\nu \leqq p_0 \leqq 0$, then the solution is given by $u(t, -x + \pi/\nu)$, where u is the solution of (2.14) with p_0 replaced by $-p_0$.) If $p_0 = 0$, many quantities in the definition of u are explicitly computable. For example, $\alpha_0 = -\pi/(2\nu), \alpha_1 = -\pi/\nu$ so that

$$A^{-1}(\eta) = -\{\arcsin(\eta/\mu) + \pi\}/\nu,$$

$$B^{-1}(\eta) = \arcsin(\eta/\mu)/\nu,$$

$$v_0(\alpha_0) = -\mu,$$

where arcsin is the principal value of the inverse of sin. We then calculate

$$\begin{aligned} D(k) &= \int_{-\mu}^{k} \{B^{-1}(\eta) - A^{-1}(\eta)\} d\eta \\ &= \frac{1}{\nu} \int_{-\mu}^{k} \left\{ 2 \arcsin\left(\frac{\eta}{\mu}\right) + \pi \right\} d\eta \\ &= \frac{1}{\nu} \left\{ 2 \left[k \arcsin\left(\frac{k}{\mu}\right) + \sqrt{\mu^2 - k^2} \right] + \pi k \right\} \end{aligned}$$

for $-\mu \leqq k \leqq 0$. In particular, $D(0) = 2\mu/\nu$, so that

$$T = \frac{\mu}{\nu} \frac{1}{a(0)c}.$$

In this case $u(T,x) \equiv 0$ so $u(t,x) \equiv 0$ for $t \geqq T$.

3. Main Theorems

We state our comparison and existence results for the equation (E).

3.1. Comparison Theorem. *Assume that conditions* (F1) *and* (F2) *hold. Assume that* (F3) *holds if F depends on the time variable t. Let u and v respectively be a sub- and supersolution of* (E)*, where Ω is a bounded open interval. If $u^* \leqq v_*$ on the parabolic boundary $\partial_p Q (= [0, T) \times \partial\Omega \cup \{0\} \times \overline{\Omega})$ of Q, then $u^* \leqq v_*$ in Q.*

3.2. Comparison Theorem for Periodic Functions. *Assume that conditions* (F1) *and* (F2) *hold. Assume that* (F3) *holds if F depends on t. Let u and v respectively be a sub- and supersolution of* (E) *in $(0, T) \times \boldsymbol{R}$. Suppose that u and v are periodic in the spatial variable x with period ϖ. If $u^* \leqq v_*$ on $\{0\} \times \boldsymbol{R}$, then $u^* \leqq v_*$ in $(0, T) \times \boldsymbol{R}$.*

3.3. Perron-Type Existence Theorem. *Assume that conditions* (F1) *and* (F2) *hold. Let u^- and u^+ respectively be a sub- and supersolution of* (E) *in $Q = (0, T) \times \Omega$, where Ω is an open interval. Suppose that $u^- \leqq u^+$ in Q and $(u^+)^* < +\infty$ and $(u^-)_* > -\infty$ in $[0, T) \times \Omega$. Then there exists a generalized solution u of* (E) *satisfying $u^- \leqq u \leqq u^+$ in Q.*

3.4. Perron-Type Existence Theorem for Periodic Functions. *Assume that conditions* (F1) *and* (F2) *hold. Let u^- and u^+ respectively be a sub- and supersolution of* (E) *in $(0, T) \times \boldsymbol{R}$. Suppose that $u^- \leqq u^+$ in $(0, T) \times \boldsymbol{R}$ and $(u^+)^* < +\infty$ and $(u^-)_* > -\infty$ in $[0, T) \times \boldsymbol{R}$. Suppose that u^- and u^+ are periodic in the spatial variable x with period ϖ. Then there exists a generalized solution u of* (E) *such that $u^- \leqq u \leqq u^+$ in $(0, T) \times \boldsymbol{R}$ and that u is periodic in x with period ϖ.*

3.5. Existence Theorem for Periodic Initial Data. *Assume that conditions* (F1) *and* (F2) *hold. Assume that* (F3) *holds if F depends on t. Suppose that $u_0 \in C(\boldsymbol{R})$ is periodic with period ϖ. Then there exists a unique function $u \in C([0, T) \times \boldsymbol{R})$ that satisfies*

(i) *u is a generalized solution of* (E) *in $(0, T) \times \boldsymbol{R}$,*

(ii) $u(0, x) = u_0(x)$ *for* $x \in \boldsymbol{R}$,

(iii) $u(t, x + \varpi) = u(t, x)$ *for* $(t, x) \in [0, T) \times \boldsymbol{R}$.

In particular, if T is arbitrary, then u can be extended globally in time.

3.6. Existence Theorem for the Dirichlet Problem. *Assume that conditions* (F1) *and* (F2) *hold. Assume that* (F3) *holds if F depends on t. Let Ω be a bounded open interval. Assume also that $F(t, p, X) = -F(t, p, -X)$. Suppose that $u_0 \in C(\overline{\Omega})$ satisfies $u_0 = 0$ on $\partial\Omega$. Then there is a unique generalized solution $u \in C([0, T) \times \overline{\Omega})$ of (E) with $u(0, x) = u_0(x)$ and $u = 0$ on $\partial\Omega$.*

3.7. Remark on the Dirichlet Problem. Theorem 3.6 applies to

$$u_t - a(u_x)\Lambda_W(u) = 0 \tag{3.1}$$

but it does not apply to

$$u_t - a(u_x)(\Lambda_W(u) - C) = 0 \tag{3.2}$$

even if C is a (nonzero) constant. This is not a technical restriction. In fact, even for smooth W with $W'' > 0$ a local solution to (3.2) may break down at the boundary $\partial\Omega$ in the sense that the gradient on $\partial\Omega$ blows up in a finite time. This phenomenon is sometimes called the *boundary-detachment phenomenon* (see, e.g., [KK]). To solve this problem globally in time we interpret the Dirichlet condition in the viscosity sense as in [KK]. We do not pursue this problem in this paper. Our global solution in Theorem 3.6 actually attains zero on $\partial\Omega$ for all time so it should not apply to (3.2).

Theorem 3.6 follows from Theorem 3.5. Indeed, we may assume that $\Omega = (0, -\varpi/2)$. We extend initial data u_0 in $(-\varpi/2, 0]$ so that $u_0(x) = -u_0(-x)$ for $x \in (-\varpi/2, 0)$. We extend u_0 in $\boldsymbol{R}$ so that it is periodic in x with period ϖ. Since $u_0 = 0$ on $\partial\Omega$, our extended u_0 is continuous. We apply Theorem 3.5 to get a unique generalized solution $u \in C([0, T) \times \boldsymbol{R})$ of (E) with $u(0, x) = u_0(x)$ and $u(t, x + \varpi) = u(t, x)$. By the symmetry assumptions on F, we see that $v(t, x) = -u(t, -x)$ satisfies (E). By Theorem 3.2 we have $u = v$, which implies that u is odd in x. In particular, $u = 0$ at $x = 0$. A similar argument shows that $u = 0$ at $x = \frac{1}{2}\varpi$. We thus observe that u satisfies the Dirichlet condition. Since u satisfies (E) in $(0, T) \times \boldsymbol{R}$ it also satisfies (E) in $(0, T) \times \Omega$ (Proposition 6.19). This property is not trivial and it will be proved in § 6. The proof of Theorem 3.6 is now complete.

3.8. Remark on Other Boundary-Value Problems. Theorem 3.5 also applies to the homogeneous Neumann problem if F and W satisfy

$$F(t, p, X) = F(t, -p, X), \quad W(p) = W(-p),$$

provided that we appropriately define the generalized solution of (E) with the Neumann condition $u_x = 0$ on $\partial\Omega$. This assertion applies to both (3.1) and (3.2) provided that $a(p) = a(-p)$ and $W(p) = W(-p)$. We shall discuss the Neumann problem in one of our forthcoming papers.

3.9. Regularity. We now examine whether or not a solution in our Existence Theorem 3.5 is regular if the initial data are regular. As observed in §2, even if the initial data are smooth, the solution may not be of class C^1 in space. However, it is Lipschitz continuous in space. A precise form is presented below. A similar property is proved in [FG] for their solution to (1.1).

3.10. Theorem on the Preservation of Lipschitz Continuity. *Let the hypotheses of Theorem 3.5 on F and u_0 hold and let u be the solution with initial data u_0. Let ν_i be either $+1$ or -1 for $i = 1, 2$ and let L be a positive constant. If*

$$\nu_1 u_0(x) \leqq \nu_1 u_0(x + \nu_2 h) + L|h|$$

for all $h > 0, x \in \boldsymbol{R}$, then

$$\nu_1 u(t, x) \leqq \nu_1 u(t, x + \nu_2 h) + L|h|$$

for all $h > 0, x \in \boldsymbol{R}, 0 \leqq t < T$. *In particular, if* u_0 *is Lipschitz continuous with constant* L, *i.e., if*

$$|u_0(x) - u_0(y)| \leqq L|x - y|$$

for all $x, y \in \boldsymbol{R}$, *then*

$$|u(t, x) - u(t, y)| \leqq L|x - y|$$

for all $x, y \in \boldsymbol{R}, 0 \leqq t < T$.

Proof. This is an easy corollary of the Comparison Theorem 3.2 since F is independent of u and x so that $u(x + \nu_2 h) \pm L|h|$ is a solution of (E) (cf. [GGIS, Corollary 2.11], [G, Theorem 3.6]). If $\nu_1 = 1$, we compare u with $v(t, x) = u(t, x + \nu_2 h) + L|h|$. Since both u and v are solutions of (E), by Theorem 3.2 we have $u(t, x) \leqq v(t, x)$. Similarly, if $\nu_1 = -1$, we compare u with $v(t, x) = u(t, x + \nu_2 h) - L|h|$ to get $u \geqq v$. $\square$

In § 4 we prepare the key maximum principle. In § 5 we introduce convolutions with faceted functions. In § 6 we give several equivalent definitions of solutions. In § 7 we prove Theorems 3.1 and 3.2 based on the results in §§ 4–6. In § 8 we prove Theorems 3.3 and 3.4. In the last section we prove Theorem 3.5 based on Theorems 3.2 and 3.4.

We are forced to assume the uniform continuity (F3) of F if F depends on t in the Comparison Theorems. We believe this restriction comes from the method. (In the proof in § 7 we have no estimates of the length of facets of test functions from below when the parameter moves.)

4. Maximum Principle

In this section we derive various maximum principles for faceted functions, which are the key tools for proving our Comparison Theorem.

A classical maximum is:

4.1. Proposition (Maximum Principle for C^2 Functions, I)**.** *Let* Ω *be a domain in* $\boldsymbol{R}^n$. *If a function* $f \in C^2(\Omega)$ *takes a local maximum over* Ω *at* $\hat{x}$, *then* $\nabla f(\hat{x}) = 0$ *and the Hessian* Hess $f(\hat{x}) \leqq 0$.

As an easy application we get

4.2. Proposition (Maximum Principle for C^2 Functions, II)**.** *Let* Ω *be an open interval. Let* $\hat{x}, \hat{y}$ *be in* Ω. *Suppose that functions* $f_1, f_2 \in C^2(\Omega)$ *and* $\theta \in C^2(\boldsymbol{R})$ *satisfy*

$$f_1(x) + f_2(y) \leqq \theta(x - y) \quad \text{for all } x \text{ and } y \in \Omega,$$
$$f_1(\hat{x}) + f_2(\hat{y}) = \theta(\hat{x} - \hat{y}).$$

Then

$$f_1''(\hat{x}) + f_2''(\hat{y}) \leqq 0.$$

This type of maximum principle for semicontinuous functions is a key tool to establish a comparison theorem for viscosity solutions of degenerate elliptic and parabolic partial differential equations [CIL]; see § G in § 7.

We extend this type of the maximum principle to faceted functions.

4.3. Theorem (Maximum Principle for Faceted Functions)**.** *Let* Ω *be an open interval. Let* $f_1, f_2 (\in C(\Omega))$ *be faceted at* $\hat{x}$*, and* $\hat{y}$*, respectively, with slope* 0 *in* Ω*. Suppose that* $\hat{x}, \hat{y} \in \Omega$ *and* $\hat{x} - \hat{y} \in I$*, where* I *is a closed interval containing* 0 (I *may be a singleton*)*. Suppose that* $\theta_0 \in C(\boldsymbol{R})$ *satisfies*

$$\theta_0 = 0 \ \text{ in } I, \quad \theta_0 > 0 \ \text{ otherwise}.$$

If functions f_1 *and* f_2 *satisfy*

$$f_1(x) + f_2(y) \leqq \theta_0(x - y) \quad \text{for all } x \text{ and } y \in \Omega,$$
$$f_1(\hat{x}) + f_2(\hat{y}) = \theta_0(\hat{x} - \hat{y}),$$

then

$$\frac{\chi(f_1, \hat{x})}{L(f_1, \hat{x})} + \frac{\chi(f_2, \hat{y})}{L(f_2, \hat{y})} \leqq 0.$$

This is not difficult to prove directly. We give a maximum principle for faceted functions depending on time, which generalizes Theorem 4.3; see Remark 4.10. Its corollary will be applied in the proof of Comparison Theorem, where we need to estimate '*time derivatives*' of functions although they are not differentiable.

4.4. Notation. We use following notational conventions in this section.
(i) For $(\hat{t}_1, \hat{x}_1), (\hat{t}_2, \hat{x}_2) \in (0, T) \times \boldsymbol{R}$, let $u_j : (0, T) \times \boldsymbol{R} \to \boldsymbol{R}$ be a uppersemicontinuous function ($j = 1, 2$) such that

$$u_j(t_j, \cdot) \in C(\boldsymbol{R}) \text{ for each } t_j \in (0, T),$$
$$u_j(\hat{t}_j, \cdot) \text{ is faceted at } \hat{x}_j \text{ in } \boldsymbol{R} \text{ with slope } 0,$$
$$L(u_j(\hat{t}_j, \cdot), \hat{x}) < \infty, \quad \text{for } j = 1, 2.$$

The faceted region $R(u_j(\hat{t}_j, \cdot), \hat{x}_j)$ is denoted by $[a_j, b_j]$, $j = 1, 2$.
(ii) Let $\theta : [0, \infty) \to [0, \infty)$ be a continuous function such that

$$\theta = 0 \text{ in } [0, \sigma] \text{ with some } \sigma > 0, \quad \theta > 0 \text{ otherwise}.$$

(iii) $S \in C((0, T) \times (0, T))$.
(iv) $\Theta(t_1, x_1, t_2, x_2) = u_1(t_1, x_1) + u_2(t_2, x_2) - \theta(|x_1 - x_2 - \hat{q}|) - S(t_1, t_2)$ with $\hat{q} = \hat{x}_1 - \hat{x}_2$.
(v) For $j = 1, 2$ we set $j' \in \{1, 2\} \setminus \{j\}$.

4.5. Theorem (Maximum Principle for Faceted Functions with Time Direction)**.** *Suppose that* $(\hat{t}_1, \hat{x}_1, \hat{t}_2, \hat{x}_2) \in (0, T) \times \boldsymbol{R} \times (0, T) \times \boldsymbol{R}$ *is a maximizer of* Θ *over* $(0, T) \times \boldsymbol{R} \times (0, T) \times \boldsymbol{R}$*, with* $\hat{q} = \hat{x}_1 - \hat{x}_2$*.*
(i)

$$\frac{\chi(u_1(\hat{t}_1, \cdot), \hat{x}_1)}{L(u_1(\hat{t}_1, \cdot), \hat{x}_1)} + \frac{\chi(u_2(\hat{t}_2, \cdot), \hat{x}_2)}{L(u_2(\hat{t}_2, \cdot), \hat{x}_2)} \leqq 0.$$

(ii) *For* $j = 1, 2$*, let* (I_j) *denote the inequality*

$$(\mathrm{I}_j) \qquad u_j(t_j, x_j) - u_j(\hat{t}_j, \hat{x}_j) \leqq S(t_1, t_2)|_{t_{j'} = \hat{t}_{j'}} - S(\hat{t}_1, \hat{t}_2).$$

Then

$$(\mathrm{I}_1)\ \textit{holds for all } (0,T)\times[a_2+\hat{q},b_2+\hat{q}],$$
$$(\mathrm{I}_2)\ \textit{holds for all } (0,T)\times[a_1-\hat{q},b_1-\hat{q}].$$

Moreover, if $\tilde{a}_j \geqq \tilde{a}_{j'}$, *then* $\chi_-(u_j(\hat{t}_j,\cdot),\hat{x}_j)=-1$ *and there exists* $\delta>0$ *such that*

$$(\mathrm{I}_j)\ \textit{holds for all } (0,T)\times[a_j-\delta,a_j),$$

and if $\tilde{b}_j \leqq \tilde{b}_{j'}$, *then* $\chi_+(u_j(\hat{t}_j,\cdot),\hat{x}_j)=-1$ *and there exists* $\delta>0$ *such that*

$$(\mathrm{I}_j)\ \textit{holds for all } (0,T)\times(b_j,b_j+\delta],$$

where

$$[\tilde{a}_1,\tilde{b}_1]=[a_1,b_1],\quad [\tilde{a}_2,\tilde{b}_2]=[a_2+\hat{q},b_2+\hat{q}].$$

4.6. Corollary. *Assume that the hypotheses of Theorem 4.5 hold. Let* Ω *be an open interval. Suppose that* $[\hat{a}_1,\hat{b}_1],[\hat{a}_2,\hat{b}_2]\subset\Omega$, *and* $\hat{x}_j\in[\hat{a}_j,\hat{b}_j]$ *for* $j=1,2$, *where*

$$[\hat{a}_1,\hat{b}_1]=[a_1,b_1]\cap[a_2+\hat{q},b_2+\hat{q}],\quad [\hat{a}_2,\hat{b}_2]=[a_1-\hat{q},b_1-\hat{q}]\cap[a_2,b_2].$$

Then there exist uppersemicontinuous functions v_1 *and* $v_2:(0,T)\times\Omega\to\boldsymbol{R}$ *such that*

(i)
$$\begin{aligned}&v_j(t_j,\cdot)\in C(\Omega)\ \textit{for each } t_j\in(0,T),\\ &v_j(\hat{t}_j,\cdot)\ \textit{is faceted at } \hat{x}_j \textit{ in } \Omega \textit{ with slope } 0 \textit{ in } \Omega,\\ &u_j\leqq v_j\ \textit{in } (0,T)\times\Omega,\\ &v_j(\hat{t}_j,\hat{x}_j)=u_j(\hat{t}_j,\hat{x}_j)\qquad \textit{for } j=1,2,\end{aligned}$$

(ii)
$$\begin{aligned}&R(v_1(\hat{t}_1,\cdot),\hat{x}_1)=[\hat{a}_1,\hat{b}_1],\qquad R(v_2(\hat{t}_2,\cdot),\hat{x}_2)=[\hat{a}_2,\hat{b}_2],\\ &L(v_1(\hat{t}_1,\cdot),\hat{x}_1)=L(v_2(\hat{t}_2,\cdot),\hat{x}_2),\end{aligned}$$

(iii)
$$\chi(v_1(\hat{t}_1,\cdot),\hat{x}_1)+\chi(v_2(\hat{t}_2,\cdot),\hat{x}_2)\leqq 0,$$

(iv) *Let* (I'_j) *denote the inequality*

$$(\mathrm{I}'_j)\qquad v_j(t_j,x_j)-v_j(\hat{t}_j,\hat{x}_j)\leqq S(t_1,t_2)|_{t_{j'}=\hat{t}_{j'}}-S(\hat{t}_1,\hat{t}_2).$$

Then

$$(\mathrm{I}'_j)\ \textit{holds for all } (0,T)\times[\hat{a}_j,\hat{b}_j]\ \textit{for } j=1,2. \tag{4.1}$$

Moreover, if $\chi_-(v_j(\hat{t}_j,\cdot),\hat{x}_j)=-1$, *then there exists* $\delta>0$ *such that*

$$(\mathrm{I}'_j)\ \textit{holds for all } (0,T)\times[\hat{a}_j-\delta,\hat{a}_j) \tag{4.2}$$

and if $\chi_+(v_j(\hat{t}_j,\cdot),\hat{x}_j)=-1$, *then there exists* $\delta>0$ *such that*

$$(\mathrm{I}'_j)\ \textit{holds for all } (0,T)\times(\hat{b}_j,\hat{b}_j+\delta]. \tag{4.3}$$

We prove Theorem 4.5 and Corollary 4.6 in several steps:

4.7. Lemma. *Assume that the hypotheses of Theorem 4.5 hold. Then*

(i) (I_1) *holds for all* $(t_1, x_1) \in (0, T) \times [a_2 + \hat{q}, b_2 + \hat{q}]$.

(ii) (I_2) *holds for all* $(t_2, x_2) \in (0, T) \times [a_1 - \hat{q}, b_1 - \hat{q}]$.

Proof. By the assumptions of the theorem, we have

$$
\begin{aligned}
& u_1(t_1, x_1) + u_2(t_2, x_2) - \theta(|x_1 - x_2 - \hat{q}|) - S(t_1, t_2) \\
& \qquad \leqq u_1(\hat{t}_1, \hat{x}_1) + u_2(\hat{t}_2, \hat{x}_2) - S(\hat{t}_1, \hat{t}_2) \\
& \qquad\qquad \text{for } (t_1, x_1), (t_2, x_2) \in (0, T) \times \boldsymbol{R}.
\end{aligned}
\tag{4.4}
$$

We take $t_2 = \hat{t}_2$ and $x_2 = x_1 - \hat{q}$ for $x_1 \in [a_2 + \hat{q}, b_2 + \hat{q}]$ in (4.4) to get

$$
\begin{aligned}
u_1(t_1, x_1) + u_2(\hat{t}_2, x_1 - \hat{q}) - S(t_1, \hat{t}_2) &\leqq u_1(\hat{t}_1, \hat{x}_1) + u_2(\hat{t}_2, \hat{x}_2) - S(\hat{t}_1, \hat{t}_2) \\
&\qquad \text{for } (t_1, x_1) \in (0, T) \times [a_2 + \hat{q}, b_2 + \hat{q}].
\end{aligned}
$$

Since $x_1 - \hat{q} \in [a_2, b_2]$, we have $u_2(\hat{t}_2, x_1 - \hat{q}) = u_2(\hat{t}_2, \hat{x}_2)$, which implies (i). Similarly, we get (ii) by substituting $t_1 = \hat{t}_1, x_1 = x_2 + \hat{q}$ for $x_2 \in R(u_1(\hat{t}_1, \cdot), \hat{x}_1) - \hat{q}$. $\square$

4.8. Lemma. *Assume that the hypotheses of Theorem 4.5 hold.*

(i) *If* $\tilde{a}_j > \tilde{a}_{j'}$, *then* $\chi_-(u_j(\hat{t}_j, \cdot), \hat{x}_j) = -1$ *and there exists* $\delta > 0$ *such that*

$$(\mathrm{I}_j) \textit{ holds for all } (t_j, x_j) \in (0, T) \times [a_j - \delta, a_j).$$

(ii) *If* $\tilde{b}_j < \tilde{b}_{j'}$, *then* $\chi_+(u_j(\hat{t}_j, \cdot), \hat{x}_j) = -1$ *and there exists* $\delta > 0$ *such that*

$$(\mathrm{I}_j) \textit{ holds for all } (t_j, x_j) \in (0, T) \times (b_j, b_j + \delta].$$

Proof. We only prove case (i), since the proof of (ii) is similar. By Lemma 4.7,

$$(\mathrm{I}_j) \text{ holds for all } (t_j, x_j) \in (0, T) \times [a_{j'} - (-1)^j \hat{q}, b_{j'} - (-1)^j \hat{q}], \text{ for } j = 1, 2.$$

Since $[a_{j'} - (-1)^j \hat{q}, b_{j'} - (-1)^j \hat{q}] \supset [a_{j'} - (-1)^j \hat{q}, a_j) = [a_j - \delta, a_j)$ with $\delta = \tilde{a}_j - \tilde{a}_{j'}$,

$$(\mathrm{I}_j) \text{ holds for all } (t_j, x_j) \in (0, T) \times [a_j - \delta, a_j).$$

It remains to prove that $\chi_-(u_j(\hat{t}_j, \cdot), \hat{x}_j) = -1$. Substituting $t_j = \hat{t}_j$ into (I_j), we have

$$u_j(\hat{t}_j, x_j) \leqq u_j(\hat{t}_j, \hat{x}_j) \text{ for all } x_j \in [a_j - \delta, a_j).$$

Since $R(u_j(\hat{t}_j, \cdot), \hat{x}_j) = [a_j, b_j]$, there exists $\eta > 0$ such that

$$u_j(\hat{t}_j, x_j) < u_j(\hat{t}_j, \hat{x}_j) \text{ for all } x_j \in [a_j - \eta, a_j),$$

which implies $\chi_-(u_j(\hat{t}_j, \cdot), \hat{x}_j) = -1$. $\square$

4.9. Lemma. *Assume that the hypotheses of Theorem 4.5 hold. Let $\sigma > 0$ be as in §4.4(ii).*

(i) *Suppose that $\tilde{a}_1 = \tilde{a}_2$; then $\chi_-(u_j(\hat{t}_j, \cdot), \hat{x}_j) = -1$ holds and*

$$(\mathrm{I}_j) \text{ holds for all } (t_j, x_j) \in (0, T) \times [a_j - \sigma, a_j) \quad \text{for } j = 1, 2.$$

(ii) *Suppose that $\tilde{b}_1 = \tilde{b}_2$; then $\chi_+(u_j(\hat{t}_j, \cdot), \hat{x}_j) = -1$ holds and*

$$(\mathrm{I}_j) \text{ holds for all } (t_j, x_j) \in (0, T) \times (b_j, b_j + \sigma] \quad \text{for } j = 1, 2.$$

Proof. We only give the proof of (i) here, since the proof of (ii) is parallel. By assumption of Theorem 4.5, we have inequality (4.4). Substituting $t_{j'} = \hat{t}_{j'}$ and $x_{j'} = a_{j'}$ in (4.4), we have

$$u_j(t_j, x_j) - \theta(|x_j - a_j|) - S(t_1, t_2)|_{t_{j'} = \hat{t}_{j'}} \leqq u_j(\hat{t}_j, \hat{x}_j) - S(\hat{t}_1, \hat{t}_2)$$
$$\text{for } (t_j, x_j) \in (0, T) \times \boldsymbol{R},$$

since $\tilde{a}_1 = \tilde{a}_2$. For $x_j \in [a_j - \sigma, a_j + \sigma]$, we have $\theta(|x_j - a_j|) = 0$, which implies that

$$(\mathrm{I}_j) \text{ holds for all } (t_j, x_j) \in (0, T) \times [a_j - \sigma, a_j).$$

Substituting $t_j = \hat{t}_j$ in (I_j), we have

$$u_j(\hat{t}_j, x_j) \leqq u_j(\hat{t}_j, \hat{x}_j) \text{ for all } x_j \in [a_j - \sigma, a_j),$$

which implies that $\chi_-(u_j(t_j, \cdot), x_j) = -1$. $\square$

Proof of Theorem 4.5. By Lemmas 4.8 and 4.9, there exist j and $k \in \{1, 2\}$ such that

$$\chi_-(u_j(\hat{t}_j, \cdot), \hat{x}_j) = -1, \quad \chi_+(u_k(\hat{t}_k, \cdot), \hat{x}_k) = -1,$$

which implies that

$$\chi(u_1(\hat{t}_1, \cdot), \hat{x}_1) + \chi(u_2(\hat{t}_2, \cdot), \hat{x}_2) \leqq 0.$$

For $(\chi(u_1(\hat{t}_1, \cdot), \hat{x}_1), \chi(u_2(\hat{t}_2, \cdot), \hat{x}_2)) = (-1, -1)$, $(-1, 0)$, $(0, 0)$ and $(0, -1)$, the validity of the inequality in (i) is trivial. We check (i) when

$$(\chi(u_1(\hat{t}_1, \cdot), \hat{x}_1), \chi(u_2(\hat{t}_2, \cdot), \hat{x}_2)) = (-1, 1).$$

Since Lemmas 4.8 and 4.9 imply that

$$R(u_1(\hat{t}_1, \cdot), \hat{x}_1) \subset R(u_2(\hat{t}_2, \cdot), \hat{x}_2) + \hat{q},$$

we have

$$L(u_1(\hat{t}_1, \cdot), \hat{x}_1) \geqq L(u_2(\hat{t}_2, \cdot), \hat{x}_2),$$

which implies (i). When $(\chi(u_1(\hat{t}_1, \cdot), \hat{x}_1), \chi(u_2(\hat{t}_2, \cdot), \hat{x}_2)) = (1, -1)$, we use a similar proof.

The property (ii) is given by Lemmas 4.7, 4.8 and 4.9. $\square$

4.10. Remark. Even if $\sigma = 0$ in Notation 4.4(ii), the assertions of Lemma 4.7 and 4.8 are still valid. If $\tilde{a}_1 = \tilde{a}_2$, then setting $t_j = \hat{t}_j$, $x_2 = x_1 - \hat{q}$ in (4.4) we have

$$\chi_-(u_j(\hat{t}_j, \cdot), \hat{x}) = -1$$

for $j = 1$ or 2. (However, the assertion for (I_j) in Lemma 4.9 may not be true.)

We observe that the proof of Theorem 4.5(i) is still valid even if $\sigma = 0$. By this remark, we note that Theorem 4.3 follows from Theorem 4.5(i) by setting

$$u_j(t, x) = f_j(x), \quad S(t_1, t_2) = 0, \quad \hat{x}_1 = \hat{x}, \quad \hat{x}_2 = \hat{y}$$

with a choice of θ in Notation 4.4(ii) such that

$$\theta(|z - \hat{q}|) \geqq \theta_0(z), \quad \text{for } z \in \boldsymbol{R}.$$

Proof of Corollary 4.6. We use the notation of Theorem 4.5. Define functions $f_{j,+}$ and $f_{j,-} \in C(\Omega)$ for $j = 1, 2$ satisfying (A) and (B).

(A) If $\tilde{a}_{j'} = \tilde{a}_j$, then $f_{j,-} \equiv 0 \equiv f_{j',-}$ in Ω, and if $\tilde{a}_{j'} < \tilde{a}_j$, then

$$f_{j,-} \equiv 0 \text{ in } \Omega, \quad f_{j',-} > 0 \text{ in } (\tilde{a}_{j'}, \tilde{a}_j), \quad f_{j',-} = 0 \text{ in } \Omega \backslash (\tilde{a}_{j'}, \tilde{a}_j).$$

(B) If $\tilde{b}_j = \tilde{b}_{j'}$, then $f_{j,+} \equiv 0 \equiv f_{j',+}$ in Ω, and if $\tilde{b}_j < \tilde{b}_{j'}$, then

$$f_{j,+} \equiv 0 \text{ in } \Omega, \quad f_{j',+} > 0 \text{ in } (\tilde{b}_j, \tilde{b}_{j'}), \quad f_{j',+} = 0 \text{ in } \Omega \backslash (\tilde{b}_j, \tilde{b}_{j'}).$$

Setting

$$v_j(t, x) = u_j(t, x) + f_{j,-}(x) + f_{j,+}(x) \quad \text{for } (t, x) \in Q \text{ and } i = 1, 2,$$

we see that (i), (ii) and (iii) hold.

Since $[\hat{a}_j, \hat{b}_j] \subset [a_{j'} - (-1)^j \hat{q}, b_{j'} - (-1)^j \hat{q}]$, we get (4.1) by Theorem 4.5 (ii). By the definition of v_1 and v_2,

$$\begin{aligned} \tilde{a}_{j'} &\leqq \tilde{a}_j \quad && \text{if } \chi_-(v_j(\hat{t}_j, \cdot), \hat{x}_j) = -1, \\ \tilde{b}_j &\leqq \tilde{b}_{j'} \quad && \text{if } \chi_+(v_j(\hat{t}_j, \cdot), \hat{x}_j) = 1, \end{aligned}$$

which imply (4.2) and (4.3). □

5. Convolutions

To regularize semicontinuous functions it is convenient to use sup- and inf-convolutions of functions. A typical way to regularize is to consider a *sup-convolution*

$$f(x, \lambda) = \sup_{\xi \in \boldsymbol{R}} \{ f(\xi) - |x - \xi|^2 / \lambda \}$$

for an (upper-semicontinuous) function f, where λ is a small positive parameter [CIL]. However, this type of convolution is not convenient for our purpose. We consider a sup-convolution with a faceted function replacing $|x - \xi|^2/\lambda$. The goal of this section is to study such sup-convolutions. A new feature of our sup-convolution is that if f assumes a local maximum at $\hat{x}$, then its sup-convolution is faceted near $\hat{x}$.

Let ϕ be a function from $\boldsymbol{R} \times (0,1]$ to $[0,\infty)$. We often assume the following conditions on ϕ.

($\Phi0$) For each $\lambda, 0 < \lambda \leqq 1, \phi(\cdot,\lambda)$ is Lipschitz continuous on every bounded set in $\boldsymbol{R}$.

($\Phi1$) $\phi(\xi,\lambda)$ is even in ξ, i.e., $\phi(\xi,\lambda) = \phi(-\xi,\lambda)$.

($\Phi2$) $\phi(\xi,\lambda)$ is nonincreasing in λ for all ξ.

($\Phi3$) $\lim\limits_{\xi\to\infty} \phi(\xi,1) = \infty$ and $\phi(\xi,\lambda)$ is nondecreasing in $\xi \geqq 0$, for $0 < \lambda \leqq 1$.

($\Phi4$) $\lim\limits_{\lambda\downarrow 0} \phi(\xi,\lambda) = \infty$ unless $\xi \not= 0$ and $\phi(0,\lambda) = 0, 0 < \lambda \leqq 1$.

($\Phi5$) Let $\sigma_\lambda = \sup\{|\xi|; \phi(\xi,\lambda) = 0\}$. Then for $0 < \lambda \leqq 1$, $\sigma_\lambda > 0$ (σ_λ is nondecreasing in λ and $\lim\limits_{\lambda\downarrow 0} \sigma_\lambda = 0$ if we assume ($\Phi2$) and ($\Phi4$)).

Let f be a function on $\boldsymbol{R}$ with values in $[-\infty,\infty)$. Then

$$f^\lambda(x) = \sup_{\xi\in\boldsymbol{R}} \{ f(\xi) - \phi(\xi - x,\lambda)\} \tag{5.1}$$

is called *a sup-convolution* of f *by* ϕ. Our assumptions in ($\Phi0$)–($\Phi4$) are rather standard. For example,

$$\phi(x,\lambda) = |x|^2/\lambda$$

satisfies ($\Phi0$)–($\Phi4$). However it does not satisfy ($\Phi5$), where $\phi(\xi,\lambda) = 0$ is assumed to be faceted at $\xi = 0$ with slope zero. Instead of this choice of ϕ we often use

$$\vartheta(x,\lambda) = \lambda\bar{\vartheta}(x/\lambda), \tag{5.2}$$

with

$$\bar{\vartheta}(x) = \begin{cases} (x-1)^2 & \text{for } x > 1, \\ 0 & \text{for } |x| < 1, \\ (x+1)^2 & \text{for } x < -1. \end{cases}$$

Clearly, ($\Phi0$)–($\Phi5$) are fulfilled for $\phi = \vartheta$. We recall fundamental properties of f^λ in (5.1) which are familiar when $\phi = |x|^2/\lambda$ [LL]. If $\phi = |x|^2/\lambda, f^\lambda$ is often used in functional analysis and is called the Yosida approximation of f. Another choice of ϕ (which does not satisfy ($\Phi5$)) was used in [IR] to study singular Hamilton-Jacobi equations.

5.1. Lemma. *Assume that* ($\Phi0$)–($\Phi4$) *hold. Let* f ($\not\equiv -\infty$) *be a function on* $\boldsymbol{R}$ *with values in* $[-\infty,\infty)$ *and assume that* f *is bounded from above on every bounded set in* $\boldsymbol{R}$ *and that*

$$\lim_{|\xi|\to\infty} \max(f(\xi),0)/\phi(\xi - x,1) = 0 \quad \textit{for each } x \in \boldsymbol{R}. \tag{5.3}$$

Let f^λ *be a sup-convolution of* f *by* ϕ. *Then*

(i) f^λ *is Lipschitz continuous on every bounded set in* $\boldsymbol{R}$.

(ii) $f^\lambda \geqq f^\mu \geqq f$ *for* $\lambda \geqq \mu > 0$ *and* $\lim_{\lambda\downarrow 0} f^\lambda(x) = f^*(x)$ *for each* $x \in \boldsymbol{R}$.

(iii) *Let* B *and* B' *be bounded open sets in* $\boldsymbol{R}$ *with* $\overline{B} \subset B'$. *Then for each* $K_0 > 0$ *there is* $\lambda_0(K_0) > 0$ *such that*

$$\sup_{x\in\overline{B}} \sup_{\xi\notin B'} H(\xi, x, \lambda) < -K_0 \quad \textit{for} \quad \lambda < \lambda_0(K_0)$$

with $H(\xi, x, \lambda) = f(\xi) - \phi(\xi - x, \lambda)$.

(a) *If* $\inf_{\overline{B}} f^* > -\infty$, *then for* $\lambda < \lambda_0' \equiv \lambda_0(\max(0, -\inf_{\overline{B}} f^*))$ *we have*

$$f^\lambda(x) = \sup_{\xi\in B'} H(\xi, x, \lambda) \quad \textit{for} \quad x \in \overline{B}.$$

(b) *If* $\hat{x}$ *is a maximizer of* f *over* B', *i.e.*, $f(\hat{x}) = \max_{B'} f$, *then* $f^\lambda(x) \leqq f(\hat{x})$ *for* $x \in \overline{B}$ *provided that*

$$\lambda < \lambda_0'' \equiv \lambda_0(\max(0, -f(\hat{x}))).$$

Proof. The proof is standard. We give it for completeness.

We may assume that f is upper-semicontinuous by replacing f by f^* in (5.1) since the value $f^\lambda(x)$ in (5.1) is unchanged with this replacement.

By (Φ2) we have $f^\lambda(x) \geqq f^\mu(x)$ for $\lambda \geqq \mu$. Since $\phi(0, \lambda) = 0$ by (Φ4), we have

$$f^\lambda(x) \geqq f(x) - \phi(x - x, \lambda) = f(x).$$

Let $\rho > 0$ be any number with

$$\rho > \overline{\rho} = \sup\{\sigma; f(x) \equiv -\infty \quad \text{for} \quad |x| < \sigma\}.$$

Then there is $\rho' > \rho$ such that

$$f^\lambda(x) = \sup_{|\xi|\leqq\rho'} H(\xi, x, \lambda) = \sup_{|\xi|\leqq\rho'} \{ f(\xi) - \phi(\xi - x, \lambda)\}, \qquad |x| \leqq \rho,\ 0 < \lambda \leqq 1. \tag{5.4}$$

Indeed, by (5.3) there is $\rho_0 > \rho$ such that

$$f(\xi) \leqq \tfrac{1}{2}\phi(\xi - x, 1) \quad \text{for } |\xi| \geqq \rho_0,\ |x| \leqq \rho,$$

so that if $|x| \leqq \rho$, then

$$f(\xi) - \phi(\xi - x, \lambda) \leqq \tfrac{1}{2}\phi(\xi - x, 1) - \phi(\xi - x, 1) = -\tfrac{1}{2}\phi(\xi - x,\ 1) \tag{5.5}$$

by (Φ2). Since there is x', $|x'| \leqq \rho$ such that $f(x') > -\infty$,

$$\begin{aligned} f^\lambda(x) &\geqq f(x') - \phi(x' - x, \lambda) \\ &\geqq f(x') - \sup_{|y|\leqq\rho} \phi(x' - y, \lambda) = M \quad \text{for } |x| \leqq \rho \end{aligned} \tag{5.6}$$

with some constant M independent of x. By (Φ3) one can take $\rho' > \rho_0$ so large that

$$-\tfrac{1}{2}\phi(\xi - x, 1) < M \quad \text{for} \quad |\xi| \geqq \rho',\ \rho \geqq |x|.$$

By (5.5) and (5.6) this implies that

$$f^\lambda(x) > f(\xi) - \phi(\xi - x, \lambda) \quad \text{for} \quad |\xi| \geqq \rho' \quad \text{and} \quad \rho \geqq |x|.$$

We have thus proved (5.4).

Since f is bounded from above on $|x| \leqq \rho'$, the supremum of (5.4) is finite, so that $f^\lambda(x)$ is finite. For each $\rho > \bar{\rho}$ we now prove that f^λ is Lipschitz continuous for $|x| \leqq \rho$. By (5.4) for each $\varepsilon > 0$ there is $\xi_0, |\xi_0| \leqq \rho'$, such that

$$f^\lambda(x) \leqq f(\xi_0) - \phi(\xi_0 - x, \lambda) + \varepsilon.$$

For $y, |y| \leqq \rho$, we have

$$\begin{aligned} f^\lambda(x) - f^\lambda(y) &\leqq f(\xi_0) - \phi(\xi_0 - x, \lambda) + \varepsilon - \{ f(\xi_0) - \phi(\xi_0 - y, \lambda)\} \\ &\leqq \phi(\xi_0 - y, \lambda) - \phi(\xi_0 - x, \lambda) + \varepsilon. \end{aligned}$$

Since $\phi(\xi, \lambda)$ is Lipschitz continuous for $|\xi| \leqq \rho + \rho'$ by (Φ0), we have

$$f^\lambda(x) - f^\lambda(y) \leqq L|x - y| + \varepsilon$$

with some $L > 0$. Sending $\varepsilon \downarrow 0$ and interchanging the role of x, y we have

$$|f^\lambda(x) - f^\lambda(y)| \leqq L|x - y| \quad \text{for } |x| \leqq \rho,\ |y| \leqq \rho.$$

We now show that

$$\lim_{\lambda \downarrow 0} f^\lambda(x) = f(x). \tag{5.7}$$

Since f is upper-semicontinuous, for each $\varepsilon > 0$ there is δ such that

$$f(\xi) \leqq f(x) + \varepsilon \quad \text{for } |\xi - x| \leqq \delta. \tag{5.8}$$

We take $\rho \geqq |x|$ and ρ' as in (5.4). Then by the monotonicity of (Φ3) we have

$$\begin{aligned} &\sup\{ f(\xi) - \phi(\xi - x, \lambda);\ |\xi| \leqq \rho',\ |\xi - x| \geqq \delta\} \\ &\leqq \sup_{|\xi| \leqq \rho'} f(\xi) - \phi(\delta, \lambda). \end{aligned} \tag{5.9}$$

By (Φ4), for each $K > 0$ there is a small $\lambda_0 > 0$ such that the right-hand side of (5.9) is dominated by $-K$ for $|\lambda| \leqq \lambda_0$. Applying (5.8) and (5.9) to (5.4) yields

$$\begin{aligned} f^\lambda(x) &\leqq \max\Big(\sup_{|\xi - x| \leqq \delta} (f(\xi) - \phi(\xi - x, \lambda)), -K\Big) \\ &\leqq \max(f(x) + \varepsilon, -K) \end{aligned}$$

if $\lambda < \lambda_0$. Take K large such that $-K < f(x)$ and fix λ_0. Then $f^\lambda(x) \leqq f(x) + \varepsilon$ for $\lambda < \lambda_0$. Since $f \leqq f^\lambda$, this implies (5.7).

It remains to prove (iii). We take $\rho\,(>\overline{\rho})$ such that $\{x \in \boldsymbol{R}; |x| \leqq \rho\}$ contains B. By the proof of (5.4) it suffices to prove that for $\rho' > \rho$,

$$\sup_{x\in\overline{B}} \sup\{H(\xi,x,\lambda); |\xi| \leqq \rho', \xi \notin B'\} \leqq -K_0 \tag{5.10}$$

for sufficiently small λ, $\lambda < \lambda_0(K_0)\,(\leqq 1)$. Let $d > 0$ be the distance from $\overline{B}$ to $\partial B'$. Then by (Φ3) with (Φ1) we have

$$H(\xi,x,\lambda) = f(\xi) - \phi(\xi - x,\lambda) \leqq f(\xi) - \phi(d,\lambda) \quad \text{for } \xi \notin B', x \in \overline{B}.$$

Since f is bounded from above on $\{\xi \in \boldsymbol{R}; |\xi| \leqq \rho'\}$, from ($\Phi$4) it now follows (5.10) with $\lambda < \lambda_0(K_0)$ provided that $\lambda_0(K_0)$ is taken so that

$$\sup_{|\xi|\leqq\rho'} f(\xi) + K_0 < \phi(d,\lambda_0(K_0)).$$

If $K_0 = \max(-\inf_{\overline{B}} f, 0)$, then $f^\lambda(x) \geqq f(x) > -K_0$ for $x \in \overline{B}$. If $\lambda < \lambda_0' = \lambda_0(K_0)$ with this K_0, then (5.10) yields

$$f^\lambda(x) > \sup\{H(\xi,x,\lambda),\ \xi \notin B',\ |\xi| \leqq \rho'\} \quad \text{for } x \in \overline{B},$$

so that

$$\begin{aligned} f^\lambda(x) &= \sup\{H(\xi,x,\lambda); |\xi| \leqq \rho'\} \quad \text{by (5.4)} \\ &= \sup\{H(\xi,x,\lambda); |\xi| \leqq \rho',\ \xi \in B\} \\ &= \sup\{H(\xi,x,\lambda); \xi \in B\} \quad \text{for } x \in \overline{B}. \end{aligned}$$

This completes the proof of (a). If $K_0 = \max(0, -f(\hat{x}))$, then for $\lambda \in \lambda_0'' = \lambda_0(K_0)$ we have, by (5.4) and (5.10), that

$$\begin{aligned} f^\lambda(x) &= \max(\sup_{\xi\in B'} H(\xi,x,\lambda), \sup\{H(\xi,x,\lambda); \xi \notin B', |\xi| \leqq \rho'\}) \\ &\leqq \max(f(\hat{x}) - 0, -K_0) \leqq f(\hat{x}) \quad \text{for} \quad x \in \overline{B}. \end{aligned}$$

The proof of (b) is now complete.

5.2. Remark. (i) The assertion of Lemma 5.1 is still valid even if $\boldsymbol{R}$ is replaced by a normed space which may have infinite dimensions provided that $\phi(\xi)$ is replaced by $\phi(|\xi|)$. The proof presented here does not depend on the local compactness of $\boldsymbol{R}$ so it applies to this case with trivial modifications.

(ii) The symmetry assumption (Φ1) is made just for convenience and it may be removed if (Φ3) is replaced by

$$\lim_{|\xi|\to\infty} \phi(\xi,1) = \infty,$$
$$\phi(\xi,\lambda) \text{ and } \phi(-\xi,\lambda) \text{ are nondecreasing in } \xi \geqq 0.$$

(iii) So far, the property (Φ5) has not been invoked. It is essentially used to prove the following Theorem, which is our main result in this section.

5.3. Theorem (Convolution with Faceted Functions). *Assume that* (Φ0)–(Φ5) *hold for* ϕ. *Assume that the hypotheses of Lemma 5.1 concerning* f *hold. Assume that* f *has a local maximum at* $\hat{x}$ *and that* f *is not a constant function. Let* f^λ *be a sup-convolution of* f *by* ϕ *defined by* (5.1). *Then there is a small* $\lambda_1, 0 < \lambda_1 \leqq 1$, *such that for* $\lambda \leqq \lambda_1$,

(i) f^λ *is faceted at* $\hat{x}$ *in* $\boldsymbol{R}$ *with slope zero and* $f^\lambda(\hat{x}) = f(\hat{x})$;
(ii) $\hat{x}$ *is an interior point of the faceted region* $R(f^\lambda, \hat{x})$.

We set

$$a = \sup\{x; f(y) = f(\hat{x}) \quad \text{for all } y \in [\hat{x}, x]\}$$

$$b = \sup\{x; f(y) \leqq f(\hat{x}) \quad \text{for all } y \in [\hat{x}, x]\} \quad (\geqq a).$$

Note that $b > \hat{x}$ if f assumes a local maximum at $\hat{x}$ and that $a < \infty$ since f is not a constant. The next lemma is a key step to prove Theorem 5.3.

5.4. Lemma. *Assume that the hypotheses of Theorem 5.3 concerning* f *and* ϕ *hold.*

(i) *Assume that* $a = b\,(> \hat{x})$. *Then, for sufficiently small* λ, *say* $\lambda < \lambda_2$, f^λ *is nondecreasing on* $[\hat{x}, b]$, *and there is* x'_λ, $\hat{x} < x'_\lambda \leqq b - \sigma_\lambda$, *such that*

$$f^\lambda(x) = f(\hat{x}) \quad \text{for all } x \in [\hat{x}, x'_\lambda],$$

$$f^\lambda(x) > f(\hat{x}) \quad \text{for all } x \in (x'_\lambda, b]$$

with σ_λ *as in* (Φ5).

(ii) *Assume that* $b > a(\geqq \hat{x})$. *Assume that* f *is upper-semicontinuous. Then for sufficiently small* λ, *say* $\lambda < \lambda_3$, *there are* y^1_λ *and* y^2_λ *with* $y^2_\lambda > y^1_\lambda > \hat{x}$, *and*

$$f^\lambda(x) = f(\hat{x}) \quad \text{for all } x \in [\hat{x}, y'_\lambda],$$

$$f^\lambda(x) < f(\hat{x}) \quad \text{for all } x \in (y^1_\lambda, y^2_\lambda).$$

Proof of Theorem 5.3 by Using Lemma 5.4. We may assume that f is upper-semicontinuous. Lemma 5.4 gives the behavior of f^λ for $x > \hat{x}$; the behavior of f^λ for $x < \hat{x}$ is obtained by Lemma 5.4 by replacing x by $-x$. Since f^λ is continuous, the behavior of f^λ so far obtained yields Theorem 5.3. We remark that Lemma 5.4 determines the value of transition numbers. It asserts that

$$\chi_+(f^\lambda, \hat{x}) = 1 \text{ if } a = b \text{ and } \chi_+(f^\lambda, \hat{x}) = -1 \text{ if } a < b. \quad \square$$

The results in Lemma 5.4 are easy to imagine since ϕ is faceted. However, the proof is not trivial although it relies only on elementary facts. Especially, the proof of part (ii) is complicated since we are forced to handle the case when f oscillates so that it takes the value $f(\hat{x})$ infinitely many times on the interval $[a, b]$.

Proof of Lemma 5.4. Since f assumes its local maximum at $\hat{x}$, there is $\delta > 0$ such that

$$f(\xi) \leqq f(\hat{x}) \text{ for } \xi, |\xi - \hat{x}| \leqq \delta.$$

Let x_0 be a point such that $\hat{x} < x_0 < b$. We apply Lemma 5.1(iii) with

$$B = (\hat{x}, x_0), \quad B' = (\hat{x} - \delta, b), \quad K_0 = \max(0, -f(\hat{x}))$$

to get

$$\sup_{\xi \notin B'} H(\xi, x, \lambda) < -K_0 \leqq f(\hat{x}) \quad \text{for all } x \in [\hat{x},\, x_0] \tag{5.11}$$

with $H(\xi, x, \lambda) = f(\xi) - \phi(\xi - x, \lambda)$, and

$$f^{\lambda}(x) \leqq f(\hat{x}) \quad \text{for all } x \in [\hat{x},\, x_0] \tag{5.12}$$

if $\lambda < \lambda_0'' \equiv \lambda_0(K_0)$, which also depends on x_0. We also apply Lemma 5.1(iii) with

$$B = (\hat{x}, b), \quad B' = (\hat{x} - \delta, b + \delta), \quad K_0 = \max(0, -\inf_{\bar{B}} f^*)$$

to get

$$f^{\lambda}(x) = \sup\{H(\xi, x, \lambda); \hat{x} - \delta \leqq \xi \leqq b + \delta\} \quad \text{for all } x \in [\hat{x},\, b] \tag{5.13}$$

provided that $\lambda < \lambda_0' \equiv \lambda_0(K_0)$ which is well-defined if $K_0 < \infty$.

(i) Since $a = b$, we see $f(x) = f(\hat{x})$ for all $x \in [\hat{x},\, b)$. This implies that

$$\inf\{\, f^*(x);\ \hat{x} \leqq x \leqq b\} = f(\hat{x}) > -\infty.$$

If $\lambda < \lambda_0'$, then (5.13) is valid. Since $f(\xi) \leqq f(\hat{x})$ for all $\xi \in [\hat{x} - \delta,\, b]$ and $\phi \geqq 0$ with $\phi(0) = 0$, we see that

$$\begin{aligned} H(\xi, x, \lambda) &= f(\xi) - \phi(\xi - x, \lambda) \leqq f(x) - \phi(x - x,\ \lambda) \\ &\leqq f(x) - 0 \quad \text{for all } \xi \in [\hat{x} - \delta,\, x] \end{aligned}$$

for all $x \in [\hat{x},\, b)$. Applying this to (5.13) we get

$$f^{\lambda}(x) = \sup\{H(\xi, x, \lambda);\ x \leqq \xi \leqq b + \delta\} \quad \text{for all } x \in [\hat{x},\, b] \tag{5.13$'$}$$

for $\lambda \leqq \lambda_0'$ since $f^{\lambda}(x) \geqq f(x) \geqq H(\xi, x, \lambda)$ for all $\xi \in [\hat{x} - \delta,\, x]$. Since $\phi(\xi, \lambda)$ is nondecreasing in $\xi \geqq 0$, we observe that

$$H(\xi, x, \lambda) = f(\xi) - \phi(\xi - x,\ \lambda) \leqq f(\xi) - \phi(\xi - y,\ \lambda)$$

for $\hat{x} \leqq x \leqq y \leqq b$ and $y \leqq \xi$. Since

$$\sup\{H(\xi, x, \lambda); x \leqq \xi \leqq y\} \leqq f(\hat{x}) = f(y) \leqq f^{\lambda}(y),$$

(5.13$'$) now yields $f^{\lambda}(x) \leqq f^{\lambda}(y)$ for $\lambda < \lambda_0'$ and $\hat{x} \leqq x \leqq y \leqq b$. We have thus proved the monotonicity: $f^{\lambda}(x) \leqq f^{\lambda}(y),\ \hat{x} \leqq x \leqq y \leqq b$ for $\lambda \leqq \lambda_0'$.

Let x_0 be fixed with $\hat{x} < x_0 < b$, and set $\lambda_2 = \min(\lambda_0', \lambda_0'')$ so that $\lambda \leqq \lambda_2$ implies (5.12). Since $f \leqq f^{\lambda}$ by Lemma 5.1(ii) and $a = b$, the estimate (5.12) yields

$$f^{\lambda}(x) = f(\hat{x}) \quad \text{for} \quad \hat{x} \leqq x \leqq x_0, \quad \lambda \leqq \lambda_2. \tag{5.14}$$

By definition of b there is a sequence $\xi_j \geqq b,\ \xi_j \downarrow b$ such that $f(\xi_j) > f(\hat{x})$. By definition of f^λ we see that

$$f^\lambda(x) \geqq f(\xi_j) - \phi(\xi_j - x,\ \lambda).$$

Assume that $b \geqq x > b - \sigma_\lambda$. Then for sufficiently large j the sequence ξ_j ($\downarrow b$) satisfies $\xi_j - x < \sigma_\lambda$. We now invoke ($\Phi$5) to get

$$\phi(\xi_j - x,\ \lambda) = 0 \quad \text{for sufficiently large } j.$$

This is the only place that (Φ5) is invoked in (i). We fix such a j and observe that

$$f^\lambda(x) \geqq f(\xi_j) - \phi(\xi_j - x, \lambda) = f(\xi_j) > f(\hat{x}).$$

In other words,

(5.15) $$f^\lambda(x) > f(\hat{x}) \quad \text{for all } x \in (b - \sigma_\lambda,\ b].$$

Since f^λ is continuous by Lemma 5.1 (i) and f^λ is nondecreasing in $[\hat{x}, b]$, the existence of x'_λ in (i) follows from (5.14) and (5.15). The proof of (i) is now complete.

(ii) We take x_0 with $a < x_0 < b$ and fix λ''_0 in (5.11) and (5.12). Since f is upper-semicontinuous, the set

$$U = (a, x_0) \backslash \{x; f(x) = f(\hat{x})\}$$

is an open set, i.e., U is a disjoint union of countably many open intervals $\{O_j\}_{j=1}^\infty$. We set

$$W_\lambda = \{\, j\,; |O_j| > 2\sigma_\lambda\},$$

where $|O_j|$ is the length of O_j. This set is at most finite. Let λ_4 be taken so that

$$2\sigma_{\lambda_4} < \max_{j \geqq 1} |O_j|,$$

which implies that W_λ is nonempty for $\lambda \leqq \lambda_4$. We set

$$x^1_\lambda = \inf\{x \in O_j;\ \ j \in W_\lambda\}.$$

Since W_λ is finite, there is a unique $j_0 \in W_\lambda$ with $x^1_\lambda = \inf O_{j_0}$. We then set

$$x^2_\lambda = \sup O_{j_0} \quad (\leqq x_0).$$

We now prove that if $\lambda \leqq \lambda_3$ with $\lambda_3 = \min(\lambda_4, \lambda''_0)$, then

(5.16) $$f^\lambda(x) = f(\hat{x}) \quad \text{for all } x \in [\hat{x}, x^1_\lambda + \sigma_\lambda],$$

(5.17) $$f^\lambda(x) < f(\hat{x}) \quad \text{for all } x \in (x^1_\lambda + \sigma_\lambda,\ x^2_\lambda - \sigma_\lambda).$$

Note that $x^1_\lambda + \sigma_\lambda < x^2_\lambda - \sigma_\lambda$ since $|O_{j_0}| > 2\sigma_\lambda$.

By definition of W_λ for all $x \in [\hat{x}, x^1_\lambda + \sigma_\lambda]$ there is ξ_0, with $x - \sigma_\lambda < \xi_0 < x + \sigma_\lambda$, such that

$$f(\xi_0) = f(\hat{x}),$$

which yields

$$f^\lambda(x) \geqq f(\xi_0) - \phi(\xi_0 - x, \lambda)$$
$$= f(\hat{x}) - 0 \qquad \text{by } (\Phi 5).$$

Property (5.16) now follows from (5.12).

It remains to prove (5.17). Assume that x fulfills $x_\lambda^1 + \sigma_\lambda < x < x_\lambda^2 - \sigma_\lambda$ and $\lambda \leqq \lambda_3$. By definition of O_{j_0} it follows that

$$H(\xi, x, \lambda) = f(\xi) - \phi(\xi - x, \lambda) < f(\hat{x}) - 0 \quad \text{for all } \xi \in (x_\lambda^1,\ x_\lambda^2). \tag{5.18}$$

Since $\lambda \leqq \lambda_3 \leqq \lambda_0''$, (5.11) yields

$$\sup\{H(\xi, x, \lambda), \xi \leqq \hat{x} - \delta \quad \text{or } \xi \geqq b\} < f(\hat{x}). \tag{5.19}$$

By definition of δ and b, if $\xi \in [\hat{x} - \delta, b]$, then $f(\xi) \leqq f(\hat{x})$ so that

$$H(\xi, x, \lambda) \leqq f(\hat{x}) - \phi(\xi - x,\ \ \lambda).$$

If ξ is outside $(x_\lambda^1,\ x_\lambda^2)$, then $\phi(\xi - x,\ \lambda) > 0$ so that

$$H(\xi, x, \lambda) < f(\hat{x}) \quad \text{for } \xi \in [\hat{x} - \delta, b] \backslash (x_\lambda', x_\lambda^2). \tag{5.20}$$

Since f is upper-semicontinuous, (5.18) and (5.20) imply that

$$\sup\{H(\xi, x, \lambda);\ \hat{x} - \delta \leqq \xi \leqq b\} < f(\hat{x}).$$

This together with (5.19) yields

$$f^\lambda(x) = \sup_{\xi \in \boldsymbol{R}} H(\xi, x, \lambda) < f(\hat{x})$$

for $x_\lambda^1 + \sigma_\lambda < x < x_\lambda^2 - \sigma_\lambda$. $\square$

In §7 we use a sup-convolution of a function by ϑ defined in (5.2). There is an advantage in using this special ϑ since it fulfills the composition rule

$$f^\beta(x) = \vartheta(x,\ \lambda - \beta) \quad \text{for } 0 < \beta < \lambda \text{ with } f(x) = \vartheta(x, \lambda). \tag{5.21}$$

We conclude this section by proving a more general composition rule which includes (5.21) as a special case. For $\rho \geqq 0$ and $\lambda > 0$ let $\vartheta(x, \rho, \lambda)$ be of the form

$$\vartheta(x, \rho, \lambda) = \begin{cases} (x - \rho)^2/\lambda, & x > \rho, \\ 0, & |x| \leqq \rho, \\ (x + \rho)^2/\lambda, & x < -\rho. \end{cases}$$

5.5. Lemma (Composition). (i) *For* $0 \leqq \alpha \leqq \rho,\ 0 < \beta < \lambda,$

$$\vartheta(x,\ \rho - \alpha,\ \lambda - \beta) = \sup_{\xi \in \boldsymbol{R}}\{\vartheta(\xi, \rho, \lambda) - \vartheta(\xi - x,\ \alpha,\ \beta)\} \quad \text{for} \quad x \in \boldsymbol{R}.$$

(ii) *For* $0 \leqq \alpha_i,\ 0 < \beta_i\ (i = 1, 2)$ *with* $\alpha_1 + \alpha_2 \leqq \rho,\ \beta_1 + \beta_2 < \lambda,$

$$\vartheta(x-y,\ \rho-(\alpha_1+\alpha_2),\ \lambda-(\beta_1+\beta_2))$$
$$= \sup_{\xi}\sup_{\eta}\{\vartheta(\xi-\eta,\ \rho,\lambda)-\vartheta(\xi-x,\ \alpha_1,\ \beta_1)-\vartheta(\eta-y,\ \alpha_2,\ \beta_2)\}$$
$$\text{for } x,y\in \boldsymbol{R}.$$

Proof. (i) By elementary calculus we can evaluate the maximum of the right-hand side to get the desired identity. We give a noncomputational proof for completeness.

We set

$$H(\xi,\ x)=\vartheta(\xi,\ \rho,\ \lambda)-\vartheta(\xi-x,\ \alpha,\ \beta),\quad f(x)=\sup_{\xi\in \boldsymbol{R}} H(\xi,\ x)$$

by suppressing parameters.

(a) If $|x|\leqq \rho-\alpha$, then $f(x)=0$. Indeed, from $\lambda>\beta,\ \rho\geqq\alpha$ it follows that

$$\vartheta(\xi-x,\ \alpha,\ \beta)\geqq \vartheta(\xi-x,\ \alpha,\ \lambda)\geqq\vartheta(\xi,\ \rho,\ \lambda),$$

which yields $H(\xi,\ x)\leqq 0$ for all $\xi\in\boldsymbol{R}$. Since $H(\xi,x)\geqq H(x,x)\geqq 0$, we now obtain $f(x)=0=\vartheta(x,\ \rho-\alpha,\ \lambda-\beta)$ for $|x|\leqq\rho-\alpha$.

(b) If $x>\rho-\alpha$, then

$$f(x)=\sup\{H(\xi,\ x);\ \xi\geqq x+\alpha\}. \tag{5.22}$$

Indeed, for $\xi,\ |\xi|\leqq x+\alpha$, we see that

$$H(\xi,\ x)=\vartheta(\xi,\ \rho,\ \lambda)-0\leqq\vartheta(x+\alpha,\ \rho,\ \lambda)-\vartheta(x+\alpha-x,\ \alpha,\beta)=H(x+\alpha,\ x).$$

For $\xi\leqq -(x+\alpha)$ we see that

$$\begin{aligned}H(\xi,\ x) &= \vartheta(\xi,\ \rho,\ \lambda)-\vartheta(x-\xi,\ \alpha,\ \beta)\\ &\leqq \vartheta(-\xi,\ \rho,\ \lambda)-\vartheta(-\xi-x,\ \alpha,\ \beta)=H(-\xi,\ x)\end{aligned}$$

since $x-\xi\geqq-\xi-x\geqq\alpha$. We thus obtain (5.22). An elementary observation shows that

$$\sup_{\xi\in\boldsymbol{R}}\left\{\frac{(\xi-\rho)^2}{\lambda}-\frac{(\xi-x-\alpha)^2}{\beta}\right\}=\sup_{\xi\geqq x+\alpha}\left\{\frac{(\xi-\rho)^2}{\lambda}-\frac{(\xi-x-\alpha)^2}{\beta}\right\}$$

since $x+\alpha>\rho,\ \lambda>\beta$. Since

$$H(\xi,\ x)=\frac{(\xi-\rho)^2}{\lambda}-\frac{(\xi-x-\alpha)^2}{\beta}\quad \text{for } \xi\geqq x+\alpha,$$

it now follows from (5.22) that

$$\begin{aligned}f(x) &= \sup_{\xi\in\boldsymbol{R}}\left\{\frac{(\xi-\rho)^2}{\lambda}-\frac{(\xi-x-\alpha)^2}{\beta}\right\}\\ &= \sup_{\eta\in\boldsymbol{R}}\left\{\frac{\eta^2}{\lambda}-\frac{(\eta-z)^2}{\beta}\right\},\quad z=x-(\rho-\alpha).\end{aligned}$$

Since $\beta < \lambda$, the last supremum is attained only at $\eta_0 = \lambda z/(\lambda - \beta)$ so that

$$f(x) = \frac{\eta_0^2}{\lambda} - \frac{(\eta_0 - z)^2}{\beta} = \frac{z^2}{\lambda - \beta}.$$

This shows that $f(x) = \vartheta(x,\ \rho - \alpha,\ \lambda - \beta)$ for $x > \rho - \alpha$. The proof for $x \leqq -(\rho - \alpha)$ is the same and so is omitted.

(ii) Using (i) twice we obtain the desired identity.

6. Equivalent Definitions of Solutions

To prove a comparison theorem (and an existence theorem) it is convenient to introduce other versions of definitions of solutions. In our original definition our test function ψ is admissible on *all* of $Q = (0, T) \times \Omega$. However, it turns out that we only need some admissibility of ψ *near* the point $(\hat{t}, \hat{x}) \in Q$ such that

$$\max_Q (u^* - \psi) = (u - \psi)(\hat{t}, \hat{x}).$$

To be precise we introduce several notions of admissible functions. In this section Ω is assumed to be a (possibly unbounded) open interval.

6.1. Definition. Let $(\hat{t}, \hat{x})$ be a point in Q. A function $\psi \in C(Q)$ is *locally admissible* near $(\hat{t}, \hat{x})$ in Q if there is a rectangular neighborhood $\hat{Q} \subset Q$ of $(\hat{t}, \hat{x})$ such that the restriction $\psi|_{\hat{Q}}$ of ψ on $\hat{Q}$ belongs to $A_p(\hat{Q})$. Since $\hat{Q}$ is rectangular, it is of form

$$\hat{Q} = J \times \hat{\Omega}$$

with open intervals J and $\hat{\Omega}$.

By definition, if $\hat{p} = \psi_x(\hat{t}, \hat{x}) \in P$, then $\psi(\hat{t}, \cdot)$ is faceted at $\hat{x}$ in $\hat{\Omega}$ with slope $\hat{p}$.

Our assumption that $\psi|_{\hat{Q}} \in A_P(\hat{Q})$ implies that $\psi(\hat{t}, \cdot)|_{\hat{\Omega}} \in C_P^2(\hat{\Omega})$. In particular, the faceted region $R(\psi(\hat{t}, \cdot), \hat{x})$ should be contained in $\hat{\Omega}$.

6.2. Semijets. We recall the definition of parabolic semijets in [CIL]. Let φ be a function on Q and $(\hat{t}, \hat{x}) \in Q$. The set of parabolic semijets is of the form

$$\begin{aligned}\mathscr{P}_Q^{2,+}\varphi(\hat{t}, \hat{x}) = \{\ (\tau, p, X) \in \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R};\ & \\ \varphi(t, x) - \varphi(\hat{t}, \hat{x}) \leqq \tau(t - \hat{t}) + p(x - \hat{x}) + \tfrac{1}{2}X(x - \hat{x})^2 & \\ + o(|t - \hat{t}| + |x - \hat{x}|^2) \quad \text{as } (t, x) \to (\hat{t}, \hat{x})\}. &\end{aligned}$$

The set $\mathscr{P}_Q^{2,-}\varphi(\hat{t}, \hat{x})$ is defined by

$$\mathscr{P}_Q^{2,-}\varphi(\hat{t}, \hat{x}) = -(\mathscr{P}_Q^{2,+}(-\varphi))(\hat{t}, \hat{x}).$$

We often write $\mathscr{P}^{\pm}$ instead of $\mathscr{P}_Q^{2,\pm}$.

If $\varphi(\hat{t}, \cdot)$ is faceted at $\hat{x}$, then the use of $\mathscr{P}^+$ does not enable us to discuss the behavior of φ near the faceted region $R(\varphi(\hat{t}, \cdot), \hat{x})$. We are interested in defining an upper time derivative on the faceted region. For this purpose we introduce semineighborhoods of a faceted region depending on the local behavior of $\varphi(\hat{t}, \cdot)$.

6.3. Semineighborhoods. Let $f \in C(\Omega)$ be faceted at x_0 with slope p_0. Let δ_+ and δ_- be (small) positive numbers. Let $R(f, x_0)$ denote the faceted region of f containing x_0. We set

$$N_+(f,\ x_0;\ \delta_+) = \{x \in \Omega;\ \sup R(f, x_0) < x \leqq \sup R(f, x_0) + \delta_+\},$$
$$N_-(f,\ x_0;\ \delta_-) = \{x \in \Omega;\ \inf R\ (f,\ x_0) - \delta_- \leqq x < \inf R(f,\ x_0)\}.$$

Our semineighborhood depends on $\chi_\pm(f, x_0)$.

(i) If $\chi_+(f,\ x_0) = \chi_-(f,\ x_0) = -1$, we set

$$\tilde{N}^{-1}(f, x_0;\ \delta_+, \delta_-) = R(f, x_0) \cup N_+(f, x_0;\ \delta_+) \cup N_-(f;\ x_0;\ \delta_-).$$

(ii) If $\chi_+(f, x_0) = 1$ and $\chi_-(f,\ x_0) = -1$, we set

$$\tilde{N}^{-1}(f,\ x_0;\ \delta_+,\ \delta_-) = R(f,\ x_0) \cup N_-(f,\ x_0;\ \delta_-).$$

(iii) If $\chi_+(f,\ x_0) = -1$ and $\chi_-(f,\ x_0) = 1$, we set

$$\tilde{N}^{-1}(x_0;\ \delta_+,\ \delta_-) = R(f, x_0) \cup N_+(f,\ x_0;\ \delta_+).$$

(iv) If $\chi_+(f,\ x_0) = \chi_-(f,\ x_0) = 1$, we set

$$\tilde{N}^{-1}(f,\ x_0;\ \delta_+,\ \delta_-) = R(f, x_0).$$

The set $\tilde{N}^{+1}(f,\ x_0;\ \delta_+,\ \delta_-)$ is defined by

$$\tilde{N}^{+1}(f,\ x_0;\ \delta_+,\ \delta_-) = \tilde{N}^{-1}(-f,\ x_0;\ \delta_+,\ \delta_-).$$

In other words, $\tilde{N}^{+1}$ is defined in the same way by interchanging 1 and -1 in (i)–(iv). We often suppress δ_+ and δ_- of $\tilde{N}^{\pm 1}$ and simply write $\tilde{N}^{\pm}(f,\ x_0)$.

6.4. Upper Time Derivatives. A function $\omega : [0, \infty) \to [0, \infty)$ is called a *modulus* if ω is a nondecreasing continuous function with $\omega(0) = 0$. For a function φ on Q we define

$$\begin{aligned} \mathscr{T}_P^+ \varphi(\hat{t}, \hat{x}) = \{\tau \in R;\ & \text{there are a modulus } \omega \text{ and three positive numbers} \\ & \delta,\ \delta_+,\ \delta_- \text{ such that} \\ & \varphi(t, x) - \varphi(\hat{t}, \hat{x}) \leqq \tau(t - \hat{t}) + p(x - \hat{x}) + \omega(|t - \hat{t}|)|t - \hat{t}| \\ & \text{for } (t, x) \in (\hat{t} - \delta,\ \hat{t} + \delta) \times \tilde{N}^{-1}(\varphi(\hat{t}, \cdot), \hat{x}, \delta_+, \delta_-)\} \end{aligned}$$

provided that $\varphi(\hat{t}, \cdot)$ is P-faceted at $\hat{x} \in \Omega$ with slope p in Ω. If $\varphi(\hat{t}, \cdot)$ is not P-faceted at $\hat{x}$, we set $\mathscr{T}_P^+ \varphi(\hat{t}, \hat{x}) = \emptyset$. The set $\mathscr{T}_P^+ \varphi(\hat{t}, \hat{x})$ is defined by

$$\mathscr{T}_P^- \varphi(\hat{t}, \hat{x}) = -(\mathscr{T}_{(-P)}^+(-\varphi))(\hat{t}, \hat{x}).$$

The set $\mathscr{T}_P^+$ is roughly the set of upper time derivatives which are uniform near the faceted region since the error term $\omega(|t - \hat{t}|)|t - \hat{t}|$ is independent of x at least on the faceted region.

6.5. Definition. Let $\varphi : Q = (0,T) \times \Omega \to \boldsymbol{R}$ be an upper-semicontinuous function. Let $(\hat{t}, \hat{x})$ be a point in Q. Assume that $\varphi(t,\cdot) \in C(\Omega)$ for t near $\hat{t}$. We say that φ is an (infinitesimally) *admissible superfunction* at $(\hat{t}, \hat{x})$ in Q if one of following conditions holds.

(A) The function $\varphi(\hat{t},\cdot)$ is P-faceted at $\hat{x}$ in Ω and $\hat{x} \in \operatorname{int} R(\varphi(\hat{t},\cdot),\hat{x})$, i.e., $\hat{x}$ is an interior point of the faceted region of $\varphi(\hat{t},\cdot)$ containing $\hat{x}$. The set $\mathscr{T}_P^{\,+}\varphi(\hat{t},\hat{x})$ is nonempty.

(B) There is $(\tau, p, X) \in \mathscr{P}^{+}\varphi(\hat{t},\hat{x})$ with $p \notin P$.

(C) The function $\varphi(\hat{t},\cdot)$ is P-faceted at $\hat{x}$ in Ω but $\hat{x} \in \partial R(\varphi(\hat{t},\cdot),\hat{x})$. The function φ is locally admissible *near* $(\hat{t},\hat{x})$ in Q.

We say φ is an *admissible subfunction* at $(\hat{t},\hat{x})$ in Q if $-\varphi$ is an admissible superfunction with P replaced by $-P$. If φ is locally admissible near $(\hat{t},\hat{x})$ in Q, it is easy to check (A), (B) or (C) so that φ is an admissible superfunction at $(\hat{t},\hat{x})$ in Q.

6.6. Definition. A real-valued function u on Q is a *subsolution in the infinitesimal sense* of (E) if $u^* < \infty$ in $[0,T) \times \overline{\Omega}$ and the following conditions are fulfilled. For each $(\hat{t},\hat{x})$ let φ be an admissible superfunction at $(\hat{t},\ \hat{x})$ in Q such that (2.2) holds. Then

(i) $\tau + F(\hat{t},\ \varphi_x(\hat{t},\hat{x}),\ \Lambda_W(\varphi(\hat{t},\cdot),\ \hat{x})) \leqq 0$ for all $\tau \in \mathscr{T}_P^{\,+}\varphi(\hat{t},\hat{x})$ if (A) in §6.5 holds;

(ii) $\tau + F(\hat{t},\ p,\ W''(p)X) \leqq 0$ for all $(\tau,p,X) \in \mathscr{P}^{+}\varphi(\hat{t},\hat{x})$ if (B) in §6.5 holds

(iii) (2.2) is valid with $\psi = \varphi$ if (C) in §6.5 holds.

The definition of *supersolution* is given by replacing u^* $(< \infty)$ by u_* $(> -\infty)$, max by min, superfunction by subfunction, $\mathscr{T}_P^{\,+}$ by $\mathscr{T}_P^{\,-}$, $\mathscr{P}^{+}$ by $\mathscr{P}^{-}$ and the inequalities in (i),(ii) and (2.2) by the opposite ones. We note that if F is continuous and W'' is continuous near p, then $\mathscr{P}^{+}$ may be replaced by $\overline{\mathscr{P}}_Q^{2,+}$, the closure of $\mathscr{P}^{+} = \mathscr{P}_Q^{2,+}$ in the sense of semijets [CIL]:

$$
\begin{aligned}
\overline{\mathscr{P}}_Q^{2,+}\varphi(\hat{t},\hat{x}) = \{(\tau,p,X);\ & \text{there are sequences } x_n \to \hat{x},\ t_n \to \hat{t},\\
& \tau_n \to \tau,\ X_n \to X \text{ satisfying } \varphi(t_n,x_n) \to \varphi(\hat{t},\hat{x}),\\
& (\tau_n,p_n,X_n) \in \mathscr{P}_Q^{2,+}\varphi(t_n,x_n),\ (t_n,x_n) \in Q\}.
\end{aligned}
$$

6.7. Definition. A real-valued function u on Q is a *subsolution in the local sense* of (E) if $u^* < \infty$ in $[0,T) \times \overline{\Omega}$ and (2.1) holds for all $(\hat{t},\hat{x}) \in Q$ and for all ψ $(\in C(Q))$ that are locally admissible near $(\hat{t},\hat{x})$ in Q and fulfill (2.2). The definition of *supersolution* is given by replacing u^* by u_*, max by min and the inequality (2.1) by the opposite one as before.

Our main goal in this section is to show that Definitions 6.6, 6.7 are equivalent to Definition 2.6.

6.8. Theorem (Local Version vs. Original Global Version). *A real-valued function on Q is a subsolution or supersolution of* (E) *if and only if it is resepectively a subsolution or supersolution in the local sense of* (E).

6.9. Theorem (Local Version vs. Infinitesimal Version). *Assume that* (F1) *holds. A real-valued function on Q is a subsolution or supersolution in the local sense of* (E) *if and only if it is respectively a subsolution or supersolution in the infinitesimal sense of* (E).

6.10. Remark. At first glance, our definition of an admissible superfunction is rather strange. In the definition of $\mathscr{T}_P^+$ we are tempted to replace $\tilde{N}^{-1}$ by $R(\varphi(\hat{t},\cdot),\ \hat{x})$. However, if we do so, then a subsolution might not be a subsolution in the infinitesimal sense. We are also tempted to replace $\tilde{N}^{-1}$ by $\tilde{N}^{+1}\cup\tilde{N}^{-1}$, a neighborhood of $R(\varphi(\hat{t},\cdot),\ \hat{x})$. This modification is good for proving Theorem 6.9. However, test functions constructed in the proof of the Comparison Theorem (§7) might not be admissible under this modification. Note that §7 is the only place the infinitesimal version of the definition is invoked. In §7 there is no situation in which $\hat{x}$ is a boundary point of $R(\varphi(\hat{t},\cdot),\ \hat{x})$ so the definition of the local version at such an $\hat{x}$ is inherited in Definitions 6.5 and 6.6.

A. Preliminary Lemmas. To prove Theorems 6.8 and 6.9 we prepare several lemmas.

6.11. Extension Lemma. *Let I be an open interval in* $(0,T)$ *and J be a bounded open interval in Ω. Assume that $\varphi\in C(Q)(Q=(0,T)\times\Omega)$ fulfills*

$$\varphi|_{I\times J}\in A_P(I\times J).$$

Let I_1 and J_1 be open intervals such that $\overline{I_1}\subset I$ and $\overline{J_1}\subset J$. Then there is a function $\psi\in A_P(Q)$ such that

$$\varphi\leqq\psi \quad \text{in } Q, \qquad \varphi=\psi \quad \text{in } \overline{I_1}\times\overline{J_1}.$$

Proof. Step 1. We prove that *if I_k and J_k (with $k=2,3$) are open intervals such that*

$$\overline{I_k}\subset I_{k+1}\ (k=1,2),\quad \overline{I_3}\subset I,$$
$$\overline{J_k}\subset J_{k+1}\ (k=1,2),\quad \overline{J_3}\subset J,$$

then there are nonnegative functions $f_2\in C(\Omega)$ and $g_2\in C(0,T)$ such that

$$f_2\equiv 0 \text{ in } J_2,\quad g_2\equiv 0 \text{ in } I_2,$$

$$\varphi(t,x)\leqq f_2(x)+g_2(t),\quad (t,x)\in Z=Q\backslash(I_3\times J_3).$$

Indeed, we take $d\in C^1(\Omega)$ that satisfies

(a) $d\equiv 0$ in J_3,

(b) $d'>0$ in the right of J_3; $d'<0$ in the left of J_3,

(c) $d(x)\to+\infty$ as x tends to the boundary of Ω.

We set

$$J(r)=\{x\in\Omega;\ d(x)<r\}$$

so that $J(0) = J_3$ and $\cup_{r \geqq 0} J(r) = \Omega$. Similarly, we take $\overline{d} \in C^1(0,T)$ that satisfies (a), (b), (c) with J_3 and Ω replaced by I_3 and $(0,T)$, respectively. We also set

$$I(r) = \{t \in (0,T);\ \overline{d}(x) < r\},$$

$$h(r) = \max\{\max(\varphi(t,x),\ 0),\ (t,x) \in Z,\ (t,x) \in \overline{I(r)} \times \overline{J(r)}\}.$$

Since $\overline{I(r)} \times \overline{J(r)}$ is compact and φ is continuous, it follows that h is continuous in $[0,\infty)$. There are nonnegative functions $f_2 \in C(\Omega)$ and $g_2 \in C(0,T)$ such that

$$f_2(x) = \begin{cases} h(d(x)) & \text{for } x \in \Omega \backslash J_3, \\ 0 & \text{for } x \in J_2, \end{cases}$$

$$g_2(t) = \begin{cases} h(\overline{d}(t)) & \text{for } t \in (0,T) \backslash I_3, \\ 0 & \text{for } t \in I_2. \end{cases}$$

By definition

$$\begin{aligned} \varphi(t,x) &\leqq h(\max(d(x),\ \overline{d}(t))) \\ &\leqq \max(f_2(x),\ g_2(t)) \\ &\leqq f_2(x) + g_2(t) \quad \text{for } (t,x) \in Z \end{aligned}$$

since both f_2 and g_2 are nonnegative. This completes the proof of Step 1.

Step 2. Since $\varphi|_{I \times J} \in A_P(I \times J)$, there is $f_1 \in C_P^2(J)$ and $g_1 \in C^1(I)$ such that

$$\varphi(t,x) = f_1(x) + g_1(t) \quad \text{in} \quad I \times J.$$

Let $\theta \in C(\Omega)$ and $\rho \in C(0,T)$ satisfy

$$0 \leqq \theta \leqq 1,\ \theta \equiv 1 \text{ in } J_3, \quad \theta \equiv 0 \text{ in a neighborhood of } \Omega \backslash J,$$
$$0 \leqq \rho \leqq 1,\ \rho \equiv 1 \text{ in } I_3, \quad \rho \equiv 0 \text{ in a neighborhood of } (0,T) \backslash I.$$

Then, it is easy to see that $f_3 = \theta\, f_1 + f_2 \in C(\Omega)$ and $g_3 = \rho\, g_1 + g_2 \in C(0,T)$ fulfills

$$f_3 = f_1 \text{ in } J_2, \quad g_3 = g_1 \text{ in } I_2,$$
$$\varphi(t,x) \leqq f_3(x) + g_3(t),\ (t,x) \in Q,$$

where f_1 and g_1 are extended by zero outside J and I respectively.

Step 3. To complete the proof it suffices to find $g \in C^1(0,T)$ and $f \in C_P^2(\Omega)$ such that $g \geqq g_3$ on $(0,T)$ with $g = g_3$ in $\overline{I_1}$ and that $f \geqq f_3$ on Ω with $f = f_3$ in $\overline{J_1}$. Since $g_3 \in C(0,T)$ fulfills $g_3 = g_1$ in I_2, so that g_3 is of class C^1 in a neighborhood of $\overline{I_1}$, it is easy to find such a function g. If $J_2 \backslash J_1$ is contained in one faceted region of f_1, we take a nonnegative C^2 function σ supported in $J_2 \backslash \overline{J_1}$ such that $(f_1 + \sigma)'$ does not belong to P at the boundary of some neighborhood J' of J_1 in J_2 and such that $f_1 + \sigma|_{J_2} \in C_P^2(J_2)$. Replacing f_3 by $f_3 + \sigma$ we may assume that $f_3 \in C(\Omega)$ fulfills

$$f_3 \in C_P^2(J_2),\ \ f_3'(x) \notin P \quad \text{for all } x \in \partial J'.$$

The proof is now complete, if we admit the next lemma, which yields a desired f.

6.12. C^2_P **Extension Lemma.** *Let J', J_2 and Ω be possibly unbounded open intervals such that $\overline{J'} \subset J_2$, $\overline{J_2} \subset \Omega$. Assume that $f_0 \in C(\Omega)$ satisfies*

$$f_0|_{J_2} \in C^2_P(J_2),\ f_0'(x) \notin P \quad \text{for all } x \in \partial J'.$$

Then there is a function $f \in C^2_P(\Omega)$ such that $f \geqq f_0$ in Ω and such that $f = f_0$ in J'. If $f_0'' \geqq 0$ in J_2, f can be taken so that $f'' \geqq 0$ in Ω.

Proof. For a sufficiently small open neighborhood J'' of $\overline{J'}$ in J_2 there is $\overline{f_0} \in C^2(\Omega)$ such that $\overline{f_0} = f_0$ in $\overline{J''}$ and $f_0 \leqq \overline{f_0}$ in Ω and that $\overline{f_0}|_{J''} \in C^2_P(J'')$ since $f_0'(x) \notin P$ near $\partial J'$. So we may assume that $f_0 \in C^2(\Omega)$ by replacing J_2 with J''. We then apply the next elementary but important lemma on intervals $\Omega \backslash J' = (b', a') \cup (a, b)$ and obtain a function f on $\Omega \backslash J'$ with $f^{(k)}(a') = f_0^{(k)}(a')$, $f^{(k)}(a) = f_0^{(k)}(a)$ $(k = 0, 1, 2)$, $f \in C^2_P(\Omega \backslash \overline{J'})$ and $f \geqq f_0$ on $\Omega \backslash J'$. If we set $f = f_0$ on $\overline{J'}$, so that $f \in C^2_P(\Omega)$, then $f \geqq f_0$ in Ω and $f = f_0$ in J'. If $f_0'' \geqq 0$ on J_2, our f satisfies $f'' \geqq 0$ in Ω as noted in the next lemma. Thus the proof of Lemma 6.12 is complete if we admit the next lemma.

6.13. Lemma. *Assume that $a < b \leqq \infty$. For $f_0 \in C^2[a, b)$ assume that $f_0'(a) \notin P$. Then there is $f \in C^2[a, b) \cap C^2_P(a, b)$ such that $f = f_0$ at $x = a$ up to second derivatives, i.e., $f^{(k)}(a) = f_0^{(k)}(a)$ $(k = 0, 1, 2)$ and $f_0 \leqq f$ in $[a, b)$. If $f_0^{(2)}(a) \geqq 0$, then f can be taken so that $f'' \geqq 0$ on $[a, b)$. The same assertion holds if we replace $[a, b)$ by $(b, a]$ when $-\infty \leqq b < a$.*

Proof. Since the proof is the same for $b > a$, we may assume that $a < b$. We may assume that $a = 0$ by a translation. We may assume that $f_0'(0) = 0$ by adding $-f_0'(0)x$ to f and replacing P by $P - f_0'(0)$.

We may also assume that there is $x_1 \geqq 0$ such that $f_0'(x) \notin P$ for $0 \leqq x \leqq x_1$ and that $f_0'' \geqq 0$ on $[x_1, b)$ with $f_0'(x_1) = 0$. Indeed, if $f_0''(0) \geqq 0$, we take $\sigma \in C[0, b)$ such that $\sigma \geqq f_0''$ in $[0, b)$, $\sigma(0) = f_0''(0)$ and $\sigma \geqq 0$. We then set

$$\overline{f_0}(x) = f_0(0) + \int_0^x dy \int_0^y dz\ \sigma(z),$$

so that $\overline{f_0} \geqq f_0$, $\overline{f_0}'' \geqq 0$ on $[0, b)$ with $f_0^{(k)}(0) = \overline{f_0}^{(k)}(0)$ $(k = 1, 2)$. In this case x_1 is chosen to be equal to zero and $f'' \geqq 0$ on $[0, b)$. If $f_0''(0) < 0$, for each $\varepsilon > 0$ we take $\sigma \in C[0, b)$ such that $\sigma \geqq f_0''$ in $[0, b)$, $\sigma(0) = f_0''(0)$ (< 0), $\sigma \geqq 0$ in $[\varepsilon, b)$, $\sigma < 0$ in $[0, \varepsilon)$ and that σ is increasing in x. Let $\overline{f_0}$ be defined as in the preceding formula for $\overline{f_0}$ but with this new σ. It turns out that if $\varepsilon > 0$ is sufficiently small, then for $\underline{p} = \inf\{\overline{f_0}'(x);\ 0 \leqq x \leqq b\}$ $(= \overline{f_0}'(\varepsilon))$ we have $[\underline{p}, 0] \cap P = \emptyset$. Moreover, there is $x_1 > \varepsilon$ such that $\overline{f_0}'(x_1) = 0$. Since $\overline{f_0}'' = \sigma \geqq 0$ on $[\varepsilon, b)$, this $\overline{f_0}$ $(\geqq f_0)$ has the required property of f_0 in this paragraph with $f_0^{(k)}(0) = \overline{f_0}^{(k)}(0)$ $(k = 1, 2)$.

It is easy to construct $h \in C^1[x_1, b)$ such that

$$h \geqq f_0'(\geqq 0),\ h' \geqq 0 \text{ on } [x_1, b),$$
$$h = f_0' \text{ near } x = x_1 \text{ with } x \geqq x_1$$

and that $\{x \in (x_1, b);\ h'(x) = p\}$ is either the empty set or a nontrivial closed interval for each $p \in P$ since f_0' is nondecreasing on $[x_1, b)$. We then set

$$f(x) = \begin{cases} f_0(x_1) + \int_{x_1}^{x} h(z)\,dz & \text{for all } x \in [x_1, b], \\ f_0(x) & \text{for all } x \in [0, x_1]. \end{cases}$$

By the choice of h this f is in $C_P^2(0, b)$ and $f \geqq f_0$ in $[0, b)$. Since $f = f_0$ near $x = 0$ (even if $x_1 = 0$), the conditions on derivatives are clearly satisfied.

B. Proof of Theorem 6.8. Since $\varphi \in A_P(Q)$ implies that φ is locally admissible near any $(\hat{t}, \hat{x})$ in Q, the 'if' part is trivial. We then prove the 'only if' part. Since the proof for a supersolution parallels that for a subsolution, we present the proof for a subsolution.

Let u be a subsolution of (E). For $(\hat{t}, \hat{x}) \in Q$ let $\varphi \in C(Q)$ be locally admissible near $(\hat{t}, \hat{x})$ in Q with

$$\max_Q (u^* - \varphi) = (u^* - \varphi)(\hat{t}, \hat{x}).$$

Case 1. $\varphi(\hat{t}, \cdot)$ *is P-faceted at $\hat{x}$*. By definition there are bounded open intervals I and J such that $\hat{t} \in I \subset (0, T)$, $R(\varphi(\hat{t}, \cdot),\ \hat{x}) \subset J \subset \Omega$ and such that $\varphi|_{I \times J} \in A_P(I \times J)$. Let I_1 and J_1 be open intervals such that $\overline{I_1} \subset I$, $\overline{J_1} \subset J$, $\hat{t} \in I_1$, and $R(\varphi(\hat{t}, \cdot), \hat{x}) \subset J_1$. By Extension Lemma 6.11 there is $\psi \in A_P(Q)$ such that $\varphi \leqq \psi$ in Q and $\varphi = \psi$ in $\overline{I_1} \times \overline{J_2}$. We thus observe that

$$\max_Q (u^* - \psi) = (u^* - \psi)(\hat{t}, \hat{x}).$$

Since u is a subsolution, we see that

$$\hat{\psi}_t + F\left(\hat{t}, \hat{\psi}_x,\ \Lambda_W(\psi(\hat{t}, \cdot),\ \hat{x})\right) \leqq 0$$

with $\hat{\psi}_t = \psi_t(\hat{t}, \hat{x})$, $\hat{\psi}_x = \psi_x(\hat{t}, \hat{x})$. Since $\varphi = \psi$ in $\overline{I_1} \times \overline{J_1}$ and J_1 contains the faceted region $R(\varphi(\hat{t}, \cdot),\ \hat{x})$, this inequality is still valid if we replace ψ by φ. We thus obtain (2.1) for φ, which we wanted to prove.

Case 2. $\varphi_x(\hat{t}, \hat{x}) \notin P$. The proof parallels that for Case 1 if we replace $R(\varphi(\hat{t}, \cdot),\ \hat{x})$ with a single point $\{\hat{x}\}$.

C. Proof of Theorem 6.9. If φ is locally admissible near $(\hat{t}, \hat{x})$ in Q, then φ is an admissible superfunction at $(\hat{t}, \hat{x})$ in Q, so the 'if' part is trivial. We then prove the 'only if' part. Since the proof for a supersolution parallels that for a subsolution, we present the proof for a subsolution.

Let u be a subsolution of (E) in the local sense. For $(\hat{t}, \hat{x}) \in Q$ let φ be an admissible superfunction at $(\hat{t}, \hat{x})$ in Q with

$$\max_Q (u^* - \varphi) = (u^* - \varphi)(\hat{t}, \hat{x}).$$

We shall construct a good locally admissible function ψ such that $\psi(\hat{t}, \hat{x}) = \varphi(\hat{t}, \hat{x})$, $\psi \geqq \varphi$ in Q.

Case 1. $\varphi(\hat{t}, \cdot)$ *is P-faceted at $\hat{x}$ in Ω and* $\hat{x} \in \operatorname{int}\ R(\varphi(\hat{t}, \cdot),\ \hat{x})$.

Step 1. By (A) of §6.5 there is $\tau \in \mathscr{T}_P^+ \varphi(\hat{t}, \hat{x})$. By definition there are a modulus ω and three positive numbers $\delta, \delta_+, \delta_-$ such that

$$\varphi(t,x) - \varphi(\hat{t},\hat{x}) \leqq \tau(t-\hat{t}) + p(x-\hat{x}) + \omega(|t-\hat{t}|)|t-\hat{t}|$$

for all $(t,x) \in Q(\delta,\delta_+,\delta_-) = (\hat{t}-\delta,\ \hat{t}+\delta) \times \tilde{N}^{-1}(\varphi(\hat{t},\cdot),\ \hat{x},\delta_+,\delta_-)$, where $p = \varphi_x(\hat{t},\hat{x}) \in P$. As is well known, there is $\theta \in C^1(\boldsymbol{R})$ such that $\theta(0) = \theta'(0) = 0$ and

$$\omega(|t-\hat{t}|)|t-\hat{t}| \leqq \theta(|t-\hat{t}|).$$

Although it is elementary, we present a general form of this fact.

6.14. Lemma. *Let ω be a modulus. Let k be a nonnegative integer. Then there is $\theta \in C^k[0,\infty)$ such that $\theta^{(j)}(0) = 0$ and $\theta^{(j)}(x) \geqq 0$ for $x \geqq 0$ with $0 \leqq j \leqq k$ and $\omega(|\rho|)|\rho|^k \leqq \theta(|\rho|)$ for all $\rho \in \boldsymbol{R}$. In particular, for a given $x_0 \in \boldsymbol{R}^m$, $G(x) = \theta(|x-x_0|)$ is C^k as a function of $x \in \boldsymbol{R}^m$.*

Proof of Lemma 6.14. This is a simple extension of [CEL, Lemma I.4]. We set

$$\theta_j(t) = \int_t^{2t} \theta_{j-1}(s)\,ds,\ j \geqq 1,\ \theta_0 = \omega \quad \text{for } t \geqq 0,$$

so that $\theta_j \in C^j[0,\infty)$ with $\theta_j^{(i)}(0) = 0$ for $0 \leqq i \leqq j$. Since θ_j is nondecreasing, we have

$$\theta_j(t) \geqq t\ \theta_{j-1}(t) \quad \text{for } t \geqq 0$$

so that $\theta_j(t) \geqq t^j\omega(t)$. We thus observe that $\theta = \theta_k$ has all desired properties; the C^k property of G at $x = x_0$ follows from $\theta^{(j)}(0) = 0$ for $0 \leqq j \leqq k$. □

By our choice of θ we have

$$\text{(6.1)} \quad \varphi(t,x) - \varphi(\hat{t},\hat{x}) \leqq p(x-\hat{x}) + g(t) \quad \text{with} \quad g(t) = \tau(t-\hat{t}) + \theta(|t-\hat{t}|)$$

for all $(t,x) \in Q(\delta,\delta_+,\delta_-)$.

Step 2. We prove: *For sufficiently small $l > 0$ there is $\xi_l \in C(\overline{J})$ with $J = \{x;\ \operatorname{dist}(x,\ R(\varphi(\hat{t},\cdot),\hat{x})) < l\}$ such that*

(i) *ξ_l is faceted at $\hat{x}$ in J with $R(\xi_l,\hat{x}) \supset R(\varphi(\hat{t},\cdot),x)$ and $\xi_l = \varphi(\hat{t},\cdot)$ in $R(\varphi(\hat{t},\cdot),\ \hat{x})$;*

(ii) *$\chi_+(\xi_l,\hat{x}) = \chi_+(\varphi(\hat{t},\cdot),\hat{x}),\ \chi_-(\xi_l,\hat{x}) = \chi_-(\varphi(\hat{t},\cdot),\ \hat{x})$;*

(iii) *$\varphi(t,x) \leqq \xi_l(x) + g(t)$ for $(t,x) \in \overline{I} \times \overline{J}$ for some neighborhood I of $\hat{t}$.*

This is the essential part of the proof, which shows that our choice of $\tilde{N}^{-1}$ in the definition of $\mathscr{T}_{\mathscr{P}}^+$ is suitable. Let $[b_-, b_+]$ denote $R(\varphi(\hat{t},\cdot),\ \hat{x})$. On (b_-, b_+) we set $\xi_l = \varphi(\hat{t},\cdot)$. We extend ξ_l on $[b_- - l, b_+ + l]$ in the following way. Since the extension on $[b_- - l,\ b_-]$ parallels that on $[b_+,\ b_+ + l]$ we only show how to extend ξ_l on $[b_+ + l]$. If $\chi_+(\varphi(\hat{t},\cdot),\ \hat{x}) = 1$, we take

$$\xi_l(x) = \sup\{\varphi(t,x) - g(t);\quad |t-\hat{t}| \leqq \tfrac{1}{2}\delta\} \quad \text{for } b_+ \leqq x \leqq b_+ + l$$

where $l < \delta_+$. Since $\varphi(t,\cdot)$ is continuous, ξ_l is lower-semicontinuous. Since φ is upper-semicontinuous, the supremum is attained. Thus we see that ξ_l is now continuous in $[b_+,\ b_+ + l]$. Since (6.1) holds for $(t,x) \in [\hat{t} - \frac{1}{2}\delta,\ \hat{t} + \frac{1}{2}\delta] \times [b_-,\ b_+]$,

$$\varphi(t,\ b_+) - g(t) \leqq p(b_+ - \hat{x}) + \varphi(\hat{t},\hat{x}) = \varphi(\hat{t}, b_+)$$

for $t \in [\hat{t} - \frac{1}{2}\delta,\ \hat{t} + \frac{1}{2}\delta]$. This implies that $\xi_l(b_+) \leqq \varphi(\hat{t},\ b_+)$. Since $\xi_l(b_+) \geqq \varphi(\hat{t},\ b_+)$ is trivial, we have

$$\xi_l(b_+) = \varphi(\hat{t}, b_+),$$

which yields the continuity of ξ_l at $x = b_+$. By definition, $\xi_l(x) \geqq \varphi(\hat{t}, x)$, so that ξ_l is a restriction on $[b_-,\ b_+ + l]$ of some faceted function $\tilde{\xi}$ with $\chi_+(\tilde{\xi},\ \hat{x}) = 1$. Since $\chi_+(\tilde{\xi},\ \hat{x})$ is independent of $\tilde{\xi}$, we safely write $\chi_+(\xi_l, \hat{x})$, which is consistent with our original χ_+ (if ξ_l is faceted at $\hat{x}$). By definition it is clear that

$$\varphi(t,x) \leqq \xi_l(x) + g(t) \quad \text{on} \quad [\hat{t} - \tfrac{1}{2}\delta,\ \hat{t} + \tfrac{1}{2}\delta] \times [b_-,\ b_+ + l].$$

Assume now that $\chi_+(\varphi(\hat{t},\cdot),\ \hat{x}) = -1$. For sufficiently small $l > 0$ we have

$$\varphi(\hat{t},\ b_+ + l) < \varphi(\hat{t},\hat{x}) + p(x - \hat{x}).$$

Since φ is upper-semicontinuous, there is a small $\sigma > 0$ ($l + \sigma < \delta_+,\ \sigma < l,\ \sigma < \frac{1}{2}\delta$) such that

$$\varphi(t,x) < \varphi(\hat{t},\hat{x}) + p(x - \hat{x})$$

for $|x - (b_+ + l)| \leqq \sigma,\ |t - \hat{t}| \leqq \sigma$. We take a continuous function $\xi_l(x)$ for $|x - (b_+ + l)| \leqq \sigma$ such that

$$\varphi(t,x) < \xi_l(x) < \varphi(\hat{t},\hat{x}) + p(x - \hat{x}),\ |x - (b_+ + l)| < \sigma,$$

$$\xi_l(x) = \varphi(\hat{t},\hat{x}) + p(x - \hat{x}) \quad \text{at } x = b_+ + l \pm \sigma.$$

If we set

$$\xi_l(x) = \varphi(\hat{t},\hat{x}) + p(x - \hat{x}) \quad \text{for } b_+ \leqq x \leqq b_+ + l - \sigma,$$

then ξ_l is continuous in $[b_+,\ b_+ + l]$ with $\chi_+(\xi_l,\ \hat{x}) = -1$. Note that (6.1) holds for

$$x \in [b_-,\ b_+ + \delta_+),\ |t - \hat{t}| < \delta \quad \text{since } \chi_+(\varphi(\hat{t},\cdot),\ \hat{x}) = -1.$$

This yields

$$\varphi(t,x) \leqq \xi_l(x) + g(t) \quad \text{for } b_+ \leqq x \leqq b_+ + l - \sigma,\ |t - \hat{t}| < \delta.$$

For x satisfying $b_+ + l - \sigma \leqq x \leqq b_+ + l$, by the choice of ξ_l we have

$$\varphi(t,x) < \xi_l(x) \quad \text{for } |t - \hat{t}| \leqq \sigma.$$

Thus (iii) holds for $|t - \hat{t}| \leqq \sigma,\ x \in [b_-,\ b_+ + l]$. Extending ξ_l on $[b_-,\ b_- - l]$ in the same way we obtain ξ_l satisfying (i), (ii), (iii) of this step.

Step 3. We prove: *There is $f_l \in C_P^2(J) \cap C(\bar{J})$ such that*

(i) *f_l is faceted at $\hat{x}$ in J with $R(f_l, \hat{x}) \subset R(\xi_l, \hat{x})$;*

(ii) $f_l = \xi_l$ *on* $R(f_l, \hat{x})$ *and* $f_l > \xi_l$ *on* $\overline{J} \backslash R(f_l, \hat{x})$;
(iii) $\chi_+(f_l, \hat{x}) = \chi_+(\xi_l, \hat{x})$, $\chi_-(f_l, \hat{x}) = \chi_-(\xi_l, \hat{x})$;
(iv) dist $(R(f_l, \hat{x}), \partial R(\xi_l, \hat{x})) \leqq l$.

It is not difficult to choose such an f_l if $C_P^2(J)$ is replaced by $C^2(J)$. Indeed, we have the next approximation lemma.

6.15. Lemma. *Let* $\xi \in C(\tilde{J})$ *be faceted at* $\hat{x}$ *in* $\tilde{J}$ *where* $\tilde{J}$ *is an open interval. For each* $l' > 0$ *there is* $f \in C^2(\tilde{J})$ *such that*

(i) f *is faceted at* $\hat{x}$ *in* $\tilde{J}$ *with* $R(f, \hat{x}) \subset R(\xi, \hat{x})$;
(ii) $f = \xi$ *on* $R(f, \hat{x})$ *and* $f > \xi$ *on* $\tilde{J} \backslash R(f, \hat{x})$;
(iii) $\chi_\pm(f, \hat{x}) = \chi_\pm(\xi, \hat{x})$;
(iv) $\mathrm{dist}(R(f, \hat{x}), \partial R(\xi, x)) \leqq l'$;
(v) $f \leqq \xi + l'$ *in* $\tilde{J}$;
(vi) $R(f, \hat{x}) = R(\xi, \hat{x})$ *if* $\chi(\xi, \hat{x}) = -1$.

To prove Step 3 we apply this lemma with $l' = l$, $\xi = \xi_l$, where ξ is continuously extended in some neighborhood $\tilde{J}$ of $\overline{J}$ and we denote f by $\overline{f_l}$. Since $\overline{f_l} \in C^2(\tilde{J})$ is faceted at $\hat{x}$, there is a neighborhood J_2 of $R(\overline{f_l}, \hat{x})$ such that $\overline{f_l}|_{J_2} \in C_P^2(J_2)$ with $\overline{f_l}'(x) \notin P$ for $x \in J_2 \backslash R(\overline{f_l}, \hat{x})$. We take a neighborhood J' of $R(\overline{f_l}, \hat{x})$ in J_2 and apply Lemma 6.12 with $f_0 = \overline{f_l}$ and obtain $f \in C_P^2(J)$, which we denote by f_l. Since $f_l = \overline{f_l}$ in J' and $f_l \geqq \overline{f_l}$ in $\tilde{J}$, our f_l satisfies all properties (i)–(iv) of this step.

Step 4. By Step 2 we observe that

$$\varphi(t, x) \leqq f_l(x) + g(t), \ (t, x) \in \overline{I} \times \overline{J}.$$

We extend $\psi_l(t, x) = f_l(x) + g(t)$ outside $\overline{I} \times \overline{J}$ so that $\psi_l \in C(Q)$ and $\varphi \leqq \psi_l$ in Q. This is possible since φ is upper-semicontinuous (cf. Lemma 6.18). Since $f_l \in C_P^2(J)$, we have $\psi_l|_{I \times J} \in A_P(I \times J)$, so that ψ_l is locally admissible near $(\hat{t}, \hat{x})$ in Q. Since $\psi_l(\hat{t}, \hat{x}) = \varphi(\hat{t}, \hat{x})$ and $\varphi \leqq \psi_l$ and since φ is a test function of u at $(\hat{t}, \hat{x})$, we have

$$\max_Q (u^* - \psi_l) = (u^* - \psi_l)(\hat{t}, \hat{x}).$$

Since u is a subsolution of (E) in the local sense,

$$g'(\hat{t}) + F(\hat{t}, \ f_l'(\hat{x}), \ \Lambda_W(f_l, \ \hat{x})) \leqq 0. \tag{6.2}$$

Note that by Steps 2 and 3, $\chi(f_l, \hat{x}) = \chi(\varphi(\hat{t}, \cdot), \ \hat{x})$ and

$$\begin{aligned}
&|L(f_l, \hat{x}) - L(\varphi(\hat{t}, \cdot), \ \hat{x})| \\
&\qquad \leqq |L(f_l, \hat{x}) - L(\xi_l, \hat{x})| + |L(\xi_l, \hat{x}) - L(\varphi(\hat{t}, \cdot), \ \hat{x})| \\
&\qquad \leqq 2l + 2l = 4l.
\end{aligned}$$

This implies that

$$\Lambda_W(f_l, \hat{x}) \to \Lambda_W(\varphi(\hat{t}, \cdot), \ \hat{x})$$

as $l \to 0$ since $f_l'(\hat{x}) = p$. By the continuity (F1) of F we send $l \to 0$ in (6.2) to get

$$\tau + F(\hat{t}, p, \Lambda_W(\varphi(\hat{t}, \cdot),\ \hat{x})) \leqq 0$$

since $g'(\hat{t}) = \tau$. We have thus proved (i) in §6.6 for Case 1.

Case 2. *There is* $(\tau, p, X) \in \mathscr{P}^+\varphi(\hat{t}, \hat{x})$ *with* $p \notin P$. By definition of $\mathscr{P}^+$ there is a modulus ω such that

$$\begin{aligned}\varphi(t,x) - \varphi(\hat{t}, \hat{x}) \leqq{}& \tau(t - \hat{t}) + p(x - \hat{x}) + \tfrac{1}{2}X(x - \hat{x})^2 \\ &+ \omega(|t - \hat{t}| + |x - x|^2)(|t - \hat{t}| + |x - \hat{x}|^2) \quad \text{in } Q.\end{aligned}$$

The last term is dominated by

$$\begin{aligned} 2\omega(2|t - \hat{t}|)|t - \hat{t}| &\quad \text{if } |t - \hat{t}| \geqq |x - \hat{x}|^2, \\ 2\omega(2|x - \hat{x}|^2)|x - \hat{x}|^2 &\quad \text{if } |t - \hat{t}| \leqq |x - \hat{x}|^2 \end{aligned}$$

so that

$$\begin{aligned}&\omega(|t - \hat{t}| + |x - \hat{x}|^2)(|t - \hat{t}| + |x - \hat{x}|^2) \\ &\qquad \leqq \omega_1(|t - \hat{t}|)|t - \hat{t}| + \omega_2(|x - \hat{x}|)|x - \hat{x}|^2\end{aligned}$$

with another modulus $\omega_1(\rho) = 2\omega(2\rho), \omega_2(\rho) = 2\omega(2\rho^2)$. By Lemma 6.14 there are $\theta_1 \in C^1[0, \infty)$, $\theta_2 \in C^2[0, \infty)$ such that $\theta_1(0) = \theta_1'(0) = 0$, $\theta_2(0) = \theta_2'(0) = \theta_2''(0) = 0$ and that

$$\begin{aligned}\omega_1(|t - \hat{t}|)|t - \hat{t}| &\leqq \theta_1(|t - \hat{t}|), \\ \omega_2(|x - \hat{x}|)|x - \hat{x}|^2 &\leqq \theta_2(|x - \hat{x}|).\end{aligned}$$

We have thus observed that

$$\begin{aligned}\varphi(t,x) - \varphi(\hat{t}, \hat{x}) \leqq{}& \tau(t - \hat{t}) + p(x - \hat{x}) + \tfrac{1}{2}X(x - \hat{x})^2 \\ &+ \theta_1(|t - \hat{t}|) + \theta_2(|x - \hat{x}|) \quad \text{in } Q.\end{aligned}$$

We then set

$$\begin{aligned} f(x) &= \varphi(\hat{t}, \hat{x}) + p(x - \hat{x}) + \tfrac{1}{2}X(x - \hat{x})^2 + \theta_2(|x - \hat{x}|), \\ g(t) &= \tau(t - \hat{t}) + \theta_1(|t - \hat{t}|), \quad \psi(t, x) = f(x) + g(t)\end{aligned}$$

so that $f \in C^2(\boldsymbol{R})$, $g \in C^1(\boldsymbol{R})$ and

$$\varphi(\hat{t}, \hat{x}) = \psi(\hat{t}, \hat{x}), \quad \varphi \leqq \psi \quad \text{in } Q.$$

This yields

$$\max_Q (u^* - \psi) = (u^* - \psi)(\hat{t}, \hat{x}),$$

since φ is a test function of u at $(\hat{t}, \hat{x})$. By definition of f and g, we have

$$\tau = g'(\hat{t}), \quad p = f'(\hat{x}) \notin P, \quad X = f''(\hat{x}).$$

Since $p \notin P,\ f \in C^2(\boldsymbol{R})$ and $g \in C^1(R)$, it follows that ψ is locally admissible near $(\hat{t}, \hat{x})$ in Q. Since u is a subsolution of (E) in the local sense, we have

$$\tau + F(\hat{t}, p,\ W''(p)X) \leqq 0,$$

which proves (ii) in §6.6 for Case 2.

Case 3. $\varphi(\hat{t}, \cdot)$ *is* P*-faceted at* $\hat{x}$ *in* Ω *but* $\hat{x} \in \partial R(\varphi(\hat{t}, \cdot), \hat{x})$. In this case, φ is locally admissible near $(\hat{t}, \hat{x})$ in Q by (C) of §6.5. Since u is a subsolution of (E) in the local sense, it is clear that (iii) of §6.6 holds. The proof of Theorem 6.9 is now complete. □

6.16. Remark. We do not use degenerate ellipticity (F2) and (F3) in proving Theorems 6.8 and 6.9. We are forced to use an approximation argument, so the continuity (F1) is invoked. In Theorem 6.8 we do not need even (F1); all we need is that F be a function from $[0, T) \times \boldsymbol{R} \times \boldsymbol{R}$ to $\boldsymbol{R}$.

6.17. Remark (Definition of Solutions). In Definition 6.7 for the local version, our test function ψ at $(\hat{t}, \hat{x})$ is assumed to be in $C(Q)$. It turns out that we may weaken this requirement so that $\psi \in C(Q')$ is locally admissible at $(\hat{t}, \hat{x})$ in Q', where Q' is some rectangular neighborhood of $(\hat{t}, \hat{x})$ in Q and that

$$\max_{Q'}(u^* - \psi) = (u^* - \psi)(\hat{t}, \hat{x}).$$

To see this, let $\hat{Q}$ be a rectangular neighborhood of $(\hat{t}, \hat{x})$ in Q' with $\psi|_{\hat{Q}} \in A_{\underline{P}}(\hat{Q})$. It suffices to prove that there is $\overline{\psi} \in C(Q)$ that satisfies

(i) $\psi = \overline{\psi}$ in $\hat{Q}$,
(ii) $\psi \leqq \overline{\psi}$ in Q',
(iii) $\max\limits_{Q'}(u^* - \psi) = \max\limits_{Q}(u^* - \overline{\psi})$.

We may assume that $\max_{Q'}(u^* - \psi) = 0$ and $u^* < \psi$ near $\partial Q'$. We may also assume that $\psi \in C(\overline{Q'})$ by taking Q' a little bit smaller. Our desired $\overline{\psi}$ is constructed by using the next lemma.

6.18. Lemma. *Let* w *be an upper-semicontinuous function in* $Q \backslash \underline{Q'}$ *with values in* $[-\infty, \infty)$. *Let* $\psi \in C(\partial Q')$ *satisfy* $w < \psi$ *on* $\partial Q'$. *Then there is* $\overline{\psi} \in C(Q \backslash Q')$ *that satisfies* $w \leqq \overline{\psi}$ *in* $Q \backslash Q'$ *and* $\overline{\psi} = \psi$ *on* $\partial Q'$.

Proof. As in the proof of Lemma 6.11, we take d and $\overline{d}$ and define $J^r = J(r)$ and $I^r = I(r)$ so that $I^0 \times J^0 = Q'$ and that $\cup_{r \geqq 0} I^r = (0, T),\ \cup_{r \geqq 0} J^r = \Omega$. We set

$$h(r) = \max\{w(t, x),\ (t, x) \in Q \backslash Q',\ (t, x) \in \overline{I^r} \times \overline{J^r}\}.$$

Since $\overline{I^r} \times \overline{J^r}$ is compact and w is upper-semicontinuous with $w < \infty$ on Q, we see that h is upper-semicontinuous in $[0, \infty)$. Since h is nondecreasing, there is a continuous function $\overline{h} \geqq h$ with $h(0) = \overline{h}(0)$. We then set

$$\psi_0(t, x) = \overline{h}(\max(d(x),\ \overline{d}(t)) \quad \text{for } (t, x) \in Q \backslash Q',$$

so that $w \leqq \underline{\psi_0}$ in $Q \backslash Q'$. Since $\sup_{\partial Q'} w < \psi$, we modify ψ_0 near $\partial Q'$ to obtain $\overline{\psi}$ satisfying $\overline{\psi} \geqq w$ in $Q \backslash Q'$ and $\overline{\psi} = \psi$ on $\partial Q'$. $\square$

From Remark 6.17 (with Theorem 6.8) it easily follows that a restriction of a subsolution is a subsolution.

6.19. Proposition (Restrictions). *Let u be a subsolution (supersolution) of* (E) *in Q. Let $Q_0 = I_0 \times J_0$ be an open rectangular set in Q_0. Then $u|_{Q_0}$ is a subsolution (supersolution) of* (E) *in Q_0.*

6.20. Proposition. *Let $\{Q_r\}_{r=1}^{\infty}$ be a sequence of rectangular domains exhausting Q, i.e., $Q_r \subset Q_{r+1}$ and $\cup_{r=1}^{\infty} Q_r = Q$. For a function u on Q, assume that u is a subsolution (supersolution) of* (E) *on each Q_r. Then u is a subsolution (supersolution) in Q provided that $u^* < \infty (u_* > -\infty)$ on $[0, T) \times \overline{\Omega}$.*

The last statement trivially follows from the definition of a subsolution in §2.

7. Proof of Comparison Theorems

The basic strategy is in finding suitable test functions of u and v to obtain a contradiction by assuming that the conclusion $u^* \leqq v_*$ were false. We use the method developed in [CGG] and [Go]. However, for example, if u and v are faceted at the points in which we are interested, the standard maximum principle [CIL] does not apply. We apply our maximum principle to overcome this difficulty. Unfortunately, u and v are not necessarily faceted, so we need to use sup-convolution to regularize these functions. Such a regularization is used in proving the standard maximum principle; however, the convolution is different from the usual one.

For $z = (t, x), z' = (s, y) \in Q = (0, T) \times \Omega$ we set

$$w(z, z') = u(z) - v(z').$$

We consider "barrier functions"

$$\Psi_\zeta(z, z'; \varepsilon, \sigma, \gamma, \gamma') = B_\varepsilon(x - y - \zeta) + S(t, s; \sigma, \gamma, \gamma'),$$
$$B_\varepsilon(x) = \tfrac{x^2}{\varepsilon},\ S(t, s; \sigma, \gamma, \gamma') = B_\sigma(t - s) + \tfrac{\gamma}{T-t} + \tfrac{\gamma'}{T-s}$$

for positive parameters $\varepsilon, \sigma, \gamma, \gamma'$ and a real parameter ζ. The term S is very large near $t = T$ or $s = T$ while $B_\varepsilon(x - y - \zeta)$ is very large away from $x - y = \zeta$ if ε is sufficiently small, and $B_\sigma(t - s)$ is very large away from $t = s$ if σ is sufficiently small. We often write $\Psi_\zeta(z, z')$ and $S(t, s)$ instead of showing the dependence on all positive parameters. As usual we shall analyze maximizers of

$$\Phi_\zeta(z, z') = w(z, z') - \Psi_\zeta(z, z').$$

In proving the Comparison Theorem we may assume that u and $-v$ are upper-semicontinuous in $\overline{Q}$ with values in $\boldsymbol{R} \cup \{-\infty\}$ by considering u^* and v_* instead of u and v. For this reason, in this section we always assume that this property holds for u and v. Since $\overline{Q}$ is compact, there is a maximizer (z_ζ, z'_ζ) in $\overline{Q} \times \overline{Q}$ of Φ_ζ over $\overline{Q} \times \overline{Q}$, i.e.,

$$\sup \Phi_\zeta \equiv \sup\{\Phi_\zeta(z, z'); (z, z') \in \overline{Q} \times \overline{Q}\} = \Phi_\zeta(z_\zeta, z'_\zeta).$$

Before going into the detail we summarize our method. We may assume that F is continuous up to $t = T$, and $u^* \leqq v_*$ on $\partial_p Q (= \partial_p \overline{Q} = [0, T] \times \partial\Omega \times \{0\} \times \overline{\Omega})$ by taking T smaller than the original T.

We first show that Φ_ζ takes on a positive maximum only on $Q \times Q$ by taking all parameters sufficiently small. This follows from the order of u and v on the parabolic boundary. The argument is standard for $\zeta = 0$. We state a quantitative version for later use (cf. §A. Choice of Parameters).

We then classify the situations depending on the derivative of Ψ_ζ at a maximizer of Φ_ζ (cf. §B). To simplify the explanation we consider the case $P = \{0\}$. If there is a sequence $\zeta_j \to 0$ such that the (spatial) derivative of Ψ_{ζ_j} at some maximizer of Φ_{ζ_j} does not equal zero, then the standard maximum principle [CIL] does apply to get a contradiction. (cf. §G). We must analyze the remaining case (Case I), i.e., the derivative of Ψ_ζ always equals zero at any maximizer $(\hat{t}_\zeta, \hat{x}_\zeta, \hat{s}_\zeta, \hat{y}_\zeta)$ for small ζ, which yields $B'_\varepsilon(\hat{x}_\zeta - \hat{y}_\zeta - \zeta) = 0$ (cf. §C). This implies that $g(\zeta) = \sup \Phi_\zeta$ is constant for small ζ (by the Constancy Lemma 7.5). From this property we see that $u(\hat{t}_0, \cdot)$ and $-v(\hat{s}_0, \cdot)$ assume their local maxima at $\hat{x}_0$ and $\hat{y}_0$ respectively (Proposition 7.7).

In general, P is not a singleton and $B'_\varepsilon(\hat{x}_\zeta - \hat{y}_\zeta - \zeta)$ is not constant as $\zeta \to 0$. Fortunately we find some ζ_0 (close to zero) such that $B'_\varepsilon(\hat{x}_\zeta - \hat{y}_\zeta - \zeta)$ is some constant $p_0 \in P$ for ζ close to ζ_0 (Lemma 7.4). The local behavior of u and v should be modified as in Proposition 7.7 so extra effort is necessary (cf. §C).

We continue to discuss the case $P = \{0\}$. The constancy of g implies that

$$u(t, x) - v(s, y) - S(t, s) \leqq u(\hat{t}_0, \hat{x}_0) - v(\hat{s}_0, \hat{y}_0) - S(\hat{t}_0, \hat{s}_0)$$

for $t, s \in (0, T)$ and x close to y, say $|x - y| < \delta$. If $u(\hat{t}_0, \cdot)$ and $v(\hat{s}_0, \cdot)$ are faceted with slope zero with facet length $< \frac{1}{2}\delta$ at $\hat{x}_0$ and $\hat{y}_0$, respectively, this inequality with $t = \hat{t}_0, s = \hat{s}_0$ implies that $\chi_u \leqq 0, \chi_v \geqq 0$, so that

$$\frac{\chi_u}{L_u} - \frac{\chi_v}{L_v} \leqq 0$$

with

$$\chi_u = \chi(u(\hat{t}_0, \cdot), \hat{x}_0), \quad L_u = L(u(\hat{t}_0, \cdot), \hat{x}_0),$$
$$\chi_v = \chi(v(\hat{s}_0, \cdot), \hat{y}_0), \quad L_v = L(v(\hat{s}_0, \cdot), \hat{y}_0).$$

Applying the infinitesimal version of the definition of solutions, we get from the above inequality for u, v and S that

$$S_t(\hat{t}_0, \hat{s}_0) + F\left(\hat{t}_0, 0, \frac{\chi_u}{L_u}\Delta\right) \leqq 0,$$
$$S_s(\hat{t}_0, \hat{s}_0) - F\left(\hat{s}_0, 0, \frac{\chi_v}{L_v}\Delta\right) \leqq 0,$$

where $\Delta = W'(+0) - W'(-0)$. Adding the last two inequalities and using the inequality $\chi_u/L_u - \chi_v/L_v \leqq 0$ and (F2), we have

$$\frac{\gamma}{(T-\hat{t}_0)^2} + \frac{\gamma'}{(T-\hat{s}_0)^2} + F\left(\hat{t}_0, 0, \frac{\chi_u}{L_u}\Delta\right) - F\left(\hat{s}_0, 0, \frac{\chi_u}{L_u}\Delta\right) \leqq 0,$$

yielding

$$(\gamma + \gamma')/T^2 \leqq \omega(|\hat{t}_0 - \hat{s}_0|)$$

where ω is a modulus of continuity of $F(t, 0, X)$ in t and where uniform continuity (F3) is invoked if F depends on t. We fix γ and γ' and letting $\sigma \to 0$ so that $\hat{t}_0 - \hat{s}_0 \to 0$ (Proposition 7.1). By the continuity of F (up to $t = T$) we end up with $(\gamma + \gamma')/T^2 \leqq 0$, a contradiction (cf. §F).

Unfortunately, u and v are not necessarily faceted, so we need to take sup-convolutions with faceted functions (cf. §D). If $u(\hat{t}_0, \cdot)$ assumes a local maximum at $\hat{x}_0$, the sup-convolution $u^\alpha(\hat{t}_0, \cdot)$ is faceted in $\boldsymbol{R}$ at $\hat{x}_0$, as we observed in §5. The length of the facet may be very large. We should apply our maximum principle in §4 to get $\chi_u/L_u - \chi_v/L_v \leqq 0$ (cf. §E). Moreover, we should be careful about the definition of solutions in the infinitesimal version as in §6.

The proof when u and v are spatially periodic is easier. We shall note the necessary alterations at the end of this section (cf. §H).

A. Choice of Parameters

7.1. Proposition. *Assume that u and $-v$ are upper-semicontinuous in $\bar{Q}$ with values in $\boldsymbol{R} \cup \{-\infty\}$. Assume that*

$$m_0 = \sup\{u(z) - v(z); z \in Q\} > 0.$$

(i) *For each m_0' $(0 < m_0' < m_0)$ there are $\gamma_0, \gamma_0' > 0$ such that*

$$\sup \Phi_\zeta > m_0' \quad \textit{for all } \varepsilon > 0, \sigma > 0, \gamma_0 > \gamma > 0, \gamma_0' > \gamma' > 0$$

and $|\zeta| \leqq \kappa_0(\varepsilon) = \frac{1}{2}(\varepsilon(m_0 - m_0'))^{1/2}$.
(ii) *Let $(z_\zeta, z_\zeta') = (t_\zeta, x_\zeta, s_\zeta, y_\zeta)$ be a maximizer of Φ_ζ over $\bar{Q} \times \bar{Q}$. Then,*

$$|t_\zeta - s_\zeta| \leqq (M\sigma)^{1/2}, \quad |x_\zeta - y_\zeta - \zeta| \leqq (M\varepsilon)^{1/2}$$

with

$$M = \sup\{w(z, z'), (z, z') \in \bar{Q} \times \bar{Q}\}$$

for all $\varepsilon > 0$, $\sigma > 0$, $\gamma_0 > \gamma > 0$, $\gamma_0' > \gamma' > 0$ and ζ with $|\zeta| \leqq \kappa_0(\varepsilon)$. In particular,

$$\lim_{\sigma \to 0} |t_\zeta - s_\zeta| = 0, \quad \lim_{\varepsilon \to 0} |x_\zeta - y_\zeta| = 0.$$

(iii) *Assume that $u \leqq v$ on $\bar{\partial}_p Q$ $(= \overline{\partial_p Q})$ and that Ω is a bounded open interval. Then there are ε_0, σ_0 such that (z_ζ, z_ζ') is an (interior) point of $Q \times Q$ for all $0 < \varepsilon < \varepsilon_0$, $0 < \sigma < \sigma_0$, $0 < \gamma < \gamma_0$, $0 < \gamma' < \gamma_0'$ and $|\zeta| \leqq \kappa_o(\varepsilon)$.*

7.2. Remark. Since w is upper-semicontinuous, we may assume in (iii) that for each $\xi > 0$

$$w(z, z') \leqq \xi, \quad z \in \overline{\partial}_p Q \text{ or } z' \in \overline{\partial}_p Q$$

for all $|x - y| < (M\varepsilon_0)^{1/2} + \kappa_0(\varepsilon_0)$, $|t - s| < (M\sigma_0)^{1/2}$ with $z = (t, x)$, $z' = (s, y)$. In what follows we assume that $m_0 > 0$ with $\xi = \frac{1}{4}m_0, m_0' = m_0 - \frac{1}{2}\xi$ and fix $\varepsilon_0, \sigma_0, \gamma_0, \gamma_0'$ so that all properties (i)–(iii) and those in Remark 7.2 hold.

Proof of Proposition 7.1. The proof is standard especially for $\zeta = 0$ cf., e.g., [GGIS], but we give it for completeness.

(i) For each $m, m_0' < m < m_0$ there is a point $z_0 = (t_0, x_0) \in Q$ satisfying

$$m_0 \geqq u(z_0) - v(z_0) \geqq m.$$

By definition we see that

$$\sup \Phi_\zeta \geqq \Phi_\zeta(z_0, z_0) \geqq m - \zeta^2/\varepsilon - S(t_0, t_0; \sigma, \gamma, \gamma').$$

Take γ_0, γ_0' so small that

$$S(t_0, t_0; \sigma, \gamma_0, \gamma_0') = \frac{\gamma_0}{T - t_0} + \frac{\gamma_0'}{T - t_0} < \frac{m - m_0'}{2}.$$

If m is close to m_0, say $m_0 - m \leqq m - m_0'$, then $\kappa_0(\varepsilon)^2 \leqq \frac{1}{2}\varepsilon(m - m_0')$. For this choice of m, we end up with

$$\sup \Phi_\zeta \geqq m - \tfrac{1}{2}(m - m_0') - \tfrac{1}{2}(m - m_0') > m_0'$$

for $\gamma < \gamma_0, \gamma' < \gamma_0'$ provided that $|\zeta| \leqq \kappa_0(\varepsilon)$.

(ii) Since $\sup \Phi_\zeta \geqq m_0' \geqq 0$, we see that $w \geqq \Psi_\zeta$ at (z_ζ, z_ζ'). Since w is upper-semicontinuous, w is bounded on $\overline{Q} \times \overline{Q}$, say $w \leqq M$. The inequality $w \geqq \Psi_\zeta$ at (z_ζ, z_ζ') yields

$$\frac{|x_\zeta - y_\zeta - \zeta|^2}{\varepsilon} \leqq M, \quad \frac{|t_\zeta - s_\zeta|^2}{\sigma} \leqq M$$

or

$$|x_\zeta - y_\zeta| \leqq (M\varepsilon)^{1/2} + |\zeta|, \quad |t_\zeta - s_\zeta| \leqq (M\sigma)^{1/2}.$$

Since M is independent of all parameters $\varepsilon, \sigma, \gamma, \gamma'$ and ξ, this yields (ii).

(iii) We argue by contradiction. Suppose that for each $0 < \gamma < \gamma_0, 0 < \gamma' < \gamma_0'$ there were a sequence $\{(\varepsilon_j, \sigma_j)\}_{j=1}^\infty$ with $\varepsilon_j \downarrow 0, \sigma_j \downarrow 0$ such that there is a maximizer

$$(z_j, z_j') \in (\overline{Q} \times \overline{Q}) \backslash (Q \times Q)$$

of $\Phi_{\zeta_j}(\cdot, \cdot; \varepsilon_j, \sigma_j, \gamma, \gamma')$ over $\overline{Q} \times \overline{Q}$ with some ζ_j, $|\zeta_j| \leqq \kappa_0(\varepsilon_j)$. Since $\overline{Q} \times \overline{Q}$ is compact, we may assume that $z_j \to \overline{z}, z_j' \to \overline{z}'$ for some point $\overline{z}, \overline{z}' \in \overline{Q}$ by taking a subsequence if necessary. By (ii) we observe that $\overline{z} = \overline{z}'$.

The condition $(z_j, z_j') \notin Q \times Q$ implies that either $z_j \in \overline{\partial}_p Q$ or $z_j' \in \overline{\partial}_p Q$ since the time component of z_j and z_j' cannot be T because of the presence of terms

$\gamma/(T-t), \gamma'/(T-s)$ in the definition of Ψ_ξ. Since $\overline{\partial}_p Q$ is closed, $\overline{z}$ belongs to $\partial_p Q$. Since $(z_j, z_j') = (t_j, x_j, s_j, y_j)$ is a maximizer of Φ_ζ, (i) implies that

$$0 < \tfrac{1}{2}m_0 < w(t_j, x_j, s_j, y_j) - B_{\varepsilon_j}(x_j - y_j - \zeta_j) - S(t_j, s_j; \sigma_j, \gamma, \gamma').$$

By the monotonic dependence on the parameters ε, σ, we see that the right-hand side is dominated from above by

$$w(t_j, x_j, s_j, y_j) - B_{\varepsilon_{j_0}}(x_j - y_j - \zeta_j) - B_{\sigma_{j_0}}(t_j - s_j)$$

if $j \geqq j_0$. Since w is upper-semicontinuous, sending $j \to \infty$ yields

$$0 < \tfrac{1}{2}m_0 \leqq w(\overline{z}, \overline{z}) = u(\overline{z}) - v(\overline{z}).$$

Since $\overline{z} \in \overline{\partial}_p Q$, this contradicts $u \leqq v$ on $\overline{\partial}_p Q$.

B. Classification

We classify the situations depending on the value of derivative of Ψ_ζ at a maximizer of Φ_ζ. Let g denote the maximum value of Φ_ζ, i.e.,

$$g(\zeta) = \sup \Phi_\zeta = \sup\{\Phi_\zeta(z, z'); (z, z') \in \overline{Q} \times \overline{Q}\}.$$

Let $\mathscr{A}(\zeta)$ denote the set of maximizers of Φ_ζ over $\overline{Q} \times \overline{Q}$, i.e.,

$$\mathscr{A}(\zeta) = \{(z, z') \in \overline{Q} \times \overline{Q}; g(\zeta) = \Phi_\zeta(z, z')\}.$$

Let $\mathscr{B}(\zeta)$ denote the set of values of derivatives $B_\varepsilon(x - y - \zeta)$ at a point of $\mathscr{A}(\zeta)$, i.e.,

$$\mathscr{B}(\zeta) = \{2(x - y - \zeta)/\varepsilon; (t, x, s, y) \in \mathscr{A}(\zeta)\}.$$

Of course, both $\mathscr{A}$ and $\mathscr{B}$ depend on $\varepsilon, \sigma, \gamma, \gamma'$ with $0 < \varepsilon < \varepsilon_0$, $0 < \sigma < \sigma_0$, $0 < \gamma < \gamma_0$, $0 < \gamma' < \gamma_0'$; however, we do not indicate its dependence since we shall fix these numbers in §§B–E. We recall basic properties of $\mathscr{A}$ and $\mathscr{B}$:

7.3. Proposition on Maximizers. (i) *The set $\mathscr{A}(\zeta)$ is a nonempty subset of $Q \times Q$ for $\zeta, |\zeta| \leqq \kappa_0(\varepsilon)$.*
(ii) *The graph of $\mathscr{A}$ (as a set-valued function) is compact, i.e.,*

$$\text{graph } \mathscr{A} = \{(\zeta, z, z'); (z, z') \in \mathscr{A}(\zeta), |\zeta| \leqq \kappa_0(\varepsilon)\}$$

is compact in $[-\kappa_0(\varepsilon), \kappa_0(\varepsilon)] \times Q \times Q$.
(iii) *The graph of $\mathscr{B}$ is compact in $[-\kappa_0(\varepsilon), \kappa_0(\varepsilon)] \times \boldsymbol{R}$.*

Proof. (i) Since w is an upper-semicontinuous function and $\overline{Q}$ is compact, $\mathscr{A}(\zeta)$ is nonempty. The assertion that $\mathscr{A}(\zeta)$ is contained in $Q \times Q$ has been proved in Proposition 7.1(iv).

(ii) We note that Φ_ζ is continuous in ζ. Since a supremum of a set of continuous functions is lower-semicontinuous, g is lower-semicontinuous so that $\Phi_\zeta - g$ is upper-semicontinuous. The set of maximizers of $\Phi_\zeta - g$ in $\Sigma = [-\kappa_0(\varepsilon), \kappa_0(\varepsilon)] \times$

$\overline{Q} \times \overline{Q}$ equals graph $\mathscr{A}$. Since $\Phi_\zeta - g$ is upper-semicontinuous on Σ, graph $\mathscr{A}$ is compact.

(iii) Since graph $\mathscr{B}$ is interpreted as the image of graph $\mathscr{A}$ of a linear mapping, graph $\mathscr{B}$ should be compact.

The situation is divided into two cases.

Case I. There is $\kappa_1 \leqq \kappa_0(\varepsilon)$ such that $\mathscr{B}(\zeta)$ is contained in P for all $\zeta, |\zeta| \leqq \kappa_1$.
Case II. This is the negation of Case I, that is, there is a sequence $\zeta_j \to 0$ such that $\mathscr{B}(\zeta_j)$ is not contained in P.

We shall study Case I in §§C–F. Our maximum principle for faceted functions will be invoked in §E.

C. Local Behavior of u and v

The next lemma is trivial if P is a singleton.

7.4. Lemma. *Consider Case I. For $0 < \kappa \leqq \kappa_1$ there are ζ_0 with $|\zeta_0| < \kappa$, $\delta > 0$ with $\delta + |\zeta_0| < \kappa$ and $p_0 \in P$ such that $\mathscr{B}(\zeta)$ contains p_0 for all ζ such that $|\zeta - \zeta_0| \leqq \delta, |\zeta| \leqq \kappa$.*

Proof. Since P is discrete and since graph $\mathscr{B}$ is closed, for each $p \in P$ the set

$$Y_p = \{\zeta;\ |\zeta| \leqq \kappa, p \in \mathscr{B}(\zeta)\}$$

is closed in $[-\kappa, \kappa]$. We know that $\mathscr{B}(\zeta)$ is a nonempty subset of P, that is,

$$[-\kappa, \kappa] = \cup\{Y_p;\ p \in P\}.$$

Since P is at most countable, the Baire category theorem [Y, Chapter 0] says that Y_{p_0} contains an interior point ζ_0 for some $p_0 \in P$. $\square$

We claim that $g(\zeta) - p_0(\zeta - \zeta_0)$ is constant on $(\zeta_0 - \delta, \zeta_0 + \delta)$. The next lemma is general and it does not need the assumptions made so far.

7.5. Constancy Lemma. *Let K be a compact set in $\boldsymbol{R}^N$ and let h be a real-valued upper-semicontinuous function on K. Let ϕ be a C^2 function on $\boldsymbol{R}^d$ with $1 \leqq d < N$. Let G be a bounded domain in $\boldsymbol{R}^d$. For each $\zeta \in G$ assume that there is a maximizer $(r_\zeta, \rho_\zeta) \in K$ of*

$$H_\zeta(r, \rho) = h(r, \rho) - \phi(r - \zeta).$$

over K such that $\nabla\phi(r_\zeta - \zeta) = 0$. Then,

$$h_\phi(\zeta) = \sup\{H_\zeta(r, \rho); (r, \rho) \in K\}$$

is constant on G.

Proof. Clearly, $H_\zeta(r_\eta, \rho_\eta) \leqq h_\phi(\zeta)$ for $\eta \in G$. By definition we have

$$H_\zeta(r_\eta, \rho_\eta) = H_\eta(r_\eta, \rho_\eta) + \phi(r_\eta - \eta) - \phi(r_\eta - \zeta),$$

which yields

$$h_\phi(\eta) \leqq h_\phi(\zeta) + \phi(r_\eta - \zeta) - \phi(r_\eta - \eta).$$

Since $\nabla\phi(r_\eta - \eta) = 0$, this yields

$$h_\phi(\eta) \leqq h_\phi(\zeta) + \frac{1}{2}\int_0^1 \nabla^2\phi(r_\eta - \eta - \tau(\zeta - \eta))d\tau(\zeta - \eta)\cdot(\zeta - \eta)$$

Since G is bounded, it follows that

$$h_\phi(\eta) - h_\phi(\zeta) \leqq C|\eta - \zeta|^2$$

with C independent of $\eta, \zeta \in G$. Changing the role of η, ζ we end up with

$$|h_\phi(\eta) - h_\phi(\zeta)| \leqq C|\eta - \zeta|^2.$$

This implies that h_ϕ is differentiable on G, whose derivative is always zero. Since G is connected, this means that h_ϕ is a constant function on G. $\square$

Remark. In Lemma 7.5 it suffices to assume that ϕ is in C^1 instead of C^2. Indeed, since $\nabla\phi$ is uniformly continuous on every compact set Z, there is a modulus ω_Z that satisfies

$$|\phi(x) - \phi(y) - \nabla\phi(y)\cdot(x-y)| \leqq \omega_Z(|x-y|)|x-y|$$

for $x, y \in Z$. Since $r_\eta - \eta$ and $r_\eta - \zeta$ move in some compact set, say Z, we see that

$$|\phi(r_\eta - \zeta) - \phi(r_\eta - \eta)| \leqq \omega_Z(|\eta - \zeta|)|\eta - \zeta|;$$

here $\nabla\phi(r_\eta - \eta) = 0$ is invoked. This yields, as in the proof of Lemma 7.5,

$$|h_\phi(\eta) - h_\phi(\zeta)| \leqq \omega_Z(|\eta - \zeta|)|\eta - \zeta|$$

which implies that h_ϕ is differentiable with $\nabla h_\phi \equiv 0$, so that h_ϕ is constant on G. $\square$

We always use the same κ, ζ_0, p_0 and δ as in Lemma 7.4. We apply Lemma 7.5 with

$$\phi(r) = B_\varepsilon(r) - p_0 r, \quad G = (\zeta_0 - \delta, \zeta_0 + \delta), \quad d = 1,$$
$$h(r,\rho) = w(t, r+y, s, y) - p_0(r - \zeta_0) - S(t,s),$$
$$\rho = (t,s,y), \quad N = 4,$$
$$K = \{(r,\rho);\ (t,s) \in [0,T]\times[0,T], \quad r = x - y, (x,y) \in \overline{\Omega}\times\overline{\Omega}\}$$

to get the constancy of $h_\phi(\zeta) = g(\zeta) - p_0(\zeta - \zeta_0)$.

7.6. Proposition. *The function $\tilde{g}(\zeta) = g(\zeta) - p_0(\zeta - \zeta_0)$ is constant on $G = (\zeta_0 - \delta, \zeta_0 + \delta)$.*

This gives information on the local behavior of w near a maximizer of Φ_{ζ_0}.

7.7. Proposition. *Let $(\hat{z}, \hat{z}') = (\hat{t}, \hat{x}, \hat{s}, \hat{y})$ be a maximizer of Φ_{ζ_0}, i.e., $(\hat{z}, \hat{z}') \in \mathscr{A}(\zeta_0)$ with the property that $p_0 = 2(\hat{x} - \hat{y} - \zeta_0)/\varepsilon$. Let u_0 and v_0 denote*

$$u_0(t,x) = u(t,x) - p_0 x, \quad v_0(s,y) = v(s,y) - p_0(y + \zeta_0).$$

Then $u_0(\hat{t},\cdot)$ and $-v_0(\hat{s},\cdot)$ take their local maxima at $\hat{x}$ and $\hat{y}$ respectively. More precisely,

$$u_0(\hat{t},x) \leqq u_0(\hat{t},\hat{x}) \quad \textit{for all } x, \quad |x-\hat{x}| < \delta, x \in \Omega,$$
$$v_0(\hat{s},y) \geqq v_0(\hat{s},\hat{y}) \quad \textit{for all } y, \quad |y-\hat{y}| < \delta, y \in \Omega.$$

Proof. Step 1. We prove: *For $w_0(z,z') = u_0(t,x) - v_0(y,s)$, let E denote*

$$E(z,z') = w_0(z,z') - S(t,s),$$

where S is the same as in the definition of Ψ_ζ. Let E_1 be

$$E_1 = \sup\{E(z,z'); \quad z=(t,x) \in \overline{Q}, \quad z'=(s,y) \in \overline{Q}, \quad (x,y) \in \Sigma_\delta\}$$

with $\Sigma_\delta = \{(x,y) \in \overline{\Omega} \times \overline{\Omega};\ |x-y-\zeta_0-q_0| < \delta\}$ with $q_0 = \hat{x}-\hat{y}-\zeta_0$. Then $E_1 = \Phi_{\zeta_0}(\hat{z},\hat{z}') - q_0^2/\varepsilon$ $(= E(\hat{z},\hat{z}'))$.

We first observe that

$$\Phi_\zeta(z,z') = w_0(z,z') - B_\varepsilon(x-y-\zeta) + p_0(x-y-\zeta_0) - S(t,s).$$

For $\zeta, |\zeta-\zeta_0| < \delta$ recalling that $p_0 = B'_\varepsilon(q_0) = 2(x_\zeta - y_\zeta - \zeta)/\varepsilon$ (independent of ζ) with some $(t_\zeta, x_\zeta, s_\zeta, y_\zeta) \in \mathscr{A}(\zeta)$ we rearrange

$$\begin{aligned}
&\varepsilon B_\varepsilon(x-y-\zeta) - \varepsilon p_0(x-y-\zeta_0)\\
&\quad = (x-y-\zeta)^2 - \varepsilon p_0(x-y-\zeta) - \varepsilon p_0(\zeta-\zeta_0)\\
&\quad = (x-y-\zeta-(x_\zeta-y_\zeta-\zeta))^2\\
&\qquad -(x_\zeta-y_\zeta-\zeta)^2 - 2(x_\zeta-y_\zeta-\zeta)(\zeta-\zeta_0)\\
&\quad = (x-y-\zeta-q_0)^2 - q_0^2 - 2q_0(\zeta-\zeta_0)
\end{aligned}$$

to obtain

$$\Phi_\zeta(z,z') = w_0(z,z') - B_\varepsilon(x-y-\zeta-q_0) - S(t,s) + q_0^2/\varepsilon + p_0(\zeta-\zeta_0).$$

Since Proposition 7.6 implies that $\tilde{g}(\zeta) = g(\zeta) - p_0(\zeta-\zeta_0)$ is constant for $|\zeta-\zeta_0| < \delta$,

$$\begin{aligned}
\Phi_{\zeta_0}(\hat{z},\hat{z}') &= \tilde{g}(\zeta_0) = \sup\{\tilde{g}(\zeta); |\zeta-\zeta_0| < \delta\}\\
&= \sup\{\Phi_\zeta(z,z') - p_0(\zeta-\zeta_0); (z,z') \in \overline{Q} \times \overline{Q}, |\zeta-\zeta_0| < \delta\}\\
&\geqq \sup_{|\zeta-\zeta_0|<\delta} \sup\{\Phi_\zeta(t,x,s,y) - p_0(\zeta-\zeta_0);\\
&\qquad\qquad x-y-q_0 = \zeta, \quad (t,x,s,y) \in \overline{Q} \times \overline{Q}\}.
\end{aligned}$$

If $x-y-q_0 = \zeta$, then

$$\begin{aligned}
\Phi_\zeta(z,z') - p_0(\zeta-\zeta_0) &= w_0(z,z') - B_\varepsilon(0) - S(t,s) + q_0^2/\varepsilon\\
&= E(z,z') + q_0^2/\varepsilon.
\end{aligned}$$

We thus obtain

$$\Phi_{\zeta_0}(\hat{z},\hat{z}') \geqq E_1 + q_0^2/\varepsilon.$$

Since $\hat{x} - \hat{y} - q_0 = \zeta_0$, we see as above that

$$\Phi_{\zeta_0}(\hat{z}, \hat{z}') = E(\hat{z}, \hat{z}') + q_0^2/\varepsilon,$$

which proves the last equality in Step 1. Since $(\hat{x}, \hat{y}) \in \Sigma_\delta$, the converse inequality

$$\Phi_{\zeta_0}(\hat{z}, \hat{z}') \leqq E_1 + q_0^2/\varepsilon$$

holds. The proof of Step 1 is now complete.

Step 2. From Step 1 it follows that

$$w_0(\hat{t}, x, \hat{s}, y) - S(\hat{t}, \hat{s}) \leqq w_0(\hat{t}, \hat{x}, \hat{s}, \hat{y}) - S(\hat{t}, \hat{s}) = E(\hat{z}, \hat{z}')$$

for $(x, y) \in \Sigma_\delta$. Setting $y = \hat{y}$ or $x = \hat{x}$ respectively yields

$$u_0(\hat{t}, x) \leqq u_0(\hat{t}, \hat{x}) \quad \text{for } |x - \hat{x}| < \delta,$$
$$v_0(\hat{s}, y) \geqq v_0(\hat{s}, \hat{y}) \quad \text{for } |y - \hat{y}| < \delta.$$

7.8. Remark. If P consists of only the zero point, Lemma 7.4 is trivial with $\zeta_0 = 0$, $\delta = \kappa$ and $p_0 = 0$. In this case $q_0 = \frac{1}{2}\varepsilon p_0 = 0, u_0 = u$, and $v_0 = v$ so that the proof of Proposition 7.7 is simplified. To understand the main idea of the proof of the Comparison Theorem it is a good idea to consider this special case.

7.9. Corollary to Proposition 7.7.

$$u_0(t, x) - v_0(s, y) - S(t, s) \leqq u_0(\hat{t}, \hat{x}) - v_0(\hat{s}, \hat{y}) - S(\hat{t}, \hat{s})$$

for all $(x, y) \in \Sigma_\delta, t, s \in [0, T]$, *where*

$$\Sigma_\delta = \{(x, y) \in \overline{\Omega} \times \overline{\Omega}; \quad |x - y - (\hat{x} - \hat{y})| < \delta\},$$
$$S(t, s) = \frac{\gamma}{T - t} + \frac{\gamma'}{T - s} + \frac{(t - s)^2}{\sigma}.$$

This follows immediately from Step 1 of the proof of Proposition 7.7.

7.10. Proposition on the Behavior Away from a Local Maximum. *Let* $(\hat{z}, \hat{z}') = (\hat{t}, \hat{x}, \hat{s}, \hat{y}), u_0$ *and* v_0 *be the same as in Proposition 7.7. Let* $\Omega = (a, b)$. *Then, there is* $x_1 \in (\hat{x}, b_1)$ *or* $y_1 \in (\hat{y}, b_2)$ *such that*

$$u_0(\hat{t}, x_1) < u_0(\hat{t}, \hat{x}) \quad \text{or } v_0(\hat{s}, y_1) > v_0(\hat{s}, \hat{y})$$

with $\eta = \hat{x} - \hat{y},\ b_1 = \min(b,\ b + \eta),\ b_2 = \min(b,\ b - \eta)$. *The same assertion is valid if* $(\hat{x}, b_1)$ *and* $(\hat{y}, b_2)$ *are replaced by* $(a_1, \hat{x})$ *and* $(a_2, \hat{y})$, *respectively, with* $a_1 = \max(a,\ a + \eta),\ a_2 = \max(a,\ a - \eta)$.

Proof. Here Remark 7.2 is explicitly invoked. We may assume that $\hat{y} \geqq \hat{x}$. Suppose that the first assertion were false. Then

$$u_0(\hat{t}, x) \geqq u_0(\hat{t}, \hat{x}) \qquad \text{for all } x \text{ with } \hat{x} < x < b + \eta,$$
$$v_0(\hat{s}, y) \leqq v_0(\hat{x}, \hat{y}) \qquad \text{for all } y \text{ with } \hat{y} < y < b.$$

Since u_0 and $-v_0$ are upper-semicontinuous on $\overline{Q}$,

$$u_0(\hat{t}, b-\eta) \geqq u_0(\hat{t}, \hat{x}), \quad v_0(\hat{s}, b) \leqq v_0(\hat{s}, \hat{y}).$$

Since $q_0 = \hat{x} - \hat{y} - \zeta_0$,

$$\begin{aligned} u_0(\hat{t}, b-\eta) - v_0(\hat{s}, b) &= u(\hat{t}, b-\eta) - p_0(b-\eta) - v(\hat{s}, b) + p_0(b+\zeta_0) \\ &= u(\hat{t}, b-\eta) - v(\hat{s}, b) + p_0(\hat{y} - \hat{x} + \zeta_0) \\ &= u(\hat{t}, b-\eta) - v(\hat{s}, b) - q_0^2/\varepsilon. \end{aligned}$$

By Remark 7.2 and Proposition 7.1(ii) this yields

$$u_0(\hat{t}, \hat{x}) - v_0(\hat{s}, \hat{y}) \leqq u_0(\hat{t}, b-\eta) - v_0(\hat{s}, b) \leqq \xi - q_0^2/\varepsilon = m_0/4 - q_0^2/\varepsilon.$$

Since $\Phi_{\zeta_0}(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geqq m_0 - \frac{1}{8}m_0$ by Proposition 7.1(i), Step 1 of the proof of Proposition 7.7 yields

$$u_0(\hat{t}, \hat{x}) - v_0(\hat{s}, \hat{y}) > E(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geqq \frac{7}{8}m_0 - \frac{q_0^2}{\varepsilon}.$$

We thus obtain a contradiction:

$$\frac{m_0}{4} - \frac{q_0^2}{\varepsilon} \geqq u_0(\hat{t}, \hat{x}) - v(\hat{s}, \hat{y}) > \frac{7}{8}m_0 - \frac{q_0^2}{\varepsilon}.$$

The proof of the second assertion is the same, and so is omitted.

D. Preparation for Applying the Maximum Principle

We fix κ in Lemma 7.4, say $\kappa = \kappa_1$. Key properties of u_0 and v_0 are summarized in Proposition 7.7, Corollary 7.9 and Proposition 7.10.

7.11. Applications of sup-convolutions. Since u_0 and v_0 in Proposition 7.7 may not be continuous, we regularize them by taking sup-convolutions introduced in §5. For $\rho \geqq 0$ and $\lambda > 0$, let $\vartheta(x, \rho, \lambda)$ denote

$$\vartheta(x, \rho, \lambda) = \begin{cases} (x-\rho)^2/\lambda, & x > \rho, \\ 0, & |x| \leqq \rho, \\ (x+\rho)^2/\lambda, & x < -\rho. \end{cases}$$

If $\rho = \lambda$ (> 0), we simply write it as $\vartheta(x, \rho)$. We consider sup-convolutions of u_0 and $-v_0$ by ϑ. For $\alpha > 0$ let u_0^α by the sup-convolution of u_0 in the x-direction, i.e.,

$$u_0^\alpha(t, x) = (u_0(t, \cdot))^\alpha = \sup\{u_0(t, \eta) - \vartheta(\eta - x, \alpha); \eta \in \boldsymbol{R}\},$$

where we use the convention that $u_0 = -\infty$ if η is outside $\overline{\Omega}$. The inf-convolution of v_0 is defined by $v_{0\beta} = -(-v_0)^\beta$ for $\beta > 0$. Both functions $u_0^\alpha, v_{0\beta}$ are defined on $[0, T] \times \boldsymbol{R}$.

7.12. Proposition. *Let $(\hat{t}, \hat{x}, \hat{s}, \hat{y}), u_0$ and v_0 be as in Proposition 7.7. Then there is $\alpha_0 > 0$ such that for $0 < \alpha \leqq \alpha_0$*

(i) $u_0^\alpha(\hat{t},\cdot)$ *and* $v_{0\alpha}(\hat{s},\cdot)$ *are respectively faceted at* $\hat{x}$ *and* $\hat{y}$ *in* $\boldsymbol{R}$ *with slope zero, and* $u_0^\alpha(\hat{t},\hat{x}) = u_0(\hat{t},\hat{x})$, $v_{0\alpha}(\hat{s},\hat{y}) = v_0(\hat{s},\hat{y})$,

(ii) *the points* $\hat{x}$ *and* $\hat{y}$ *respectively belong to the interior of the faceted region* $R(u_0^\alpha(\hat{t},\cdot),\hat{x})$ *and* $R(v_{0\alpha}(\hat{s},\cdot),\hat{y})$,

(iii)
$$\hat{y} \in \{R(u_0^\alpha(\hat{t},\cdot),\hat{x}) - \eta\} \cap R(v_{0\alpha}(\hat{s},\cdot),\hat{y}) \subset \Omega,$$
$$\hat{x} \in R(u_0^\alpha(\hat{t},\cdot),x) \cap \{R(v_{0\alpha}(\hat{s},\cdot),\hat{y}) + \eta\} \subset \Omega \quad \textit{with } \eta = \hat{x} - \hat{y}.$$

Proof. By Lemma 5.1, convolution u_0^α is locally Lipschitz continuous in the space variable. Since $u_0(\hat{t},\cdot)$ assumes its local maximum at $\hat{x}$, by Theorem 5.3 on convolution with faceted functions, $u_0^\alpha(\hat{t},\cdot)$ is faceted at x in $\boldsymbol{R}$ for small α and $u_0^\alpha(\hat{t},\cdot)$ is constant $u_0(\hat{t},\hat{x})$ around $\hat{x}$ from a fundamental property of our sup-convolutions; note that u_0 and $-v_0$ are bounded from above. This proves (i) and (ii) for u_0^α. The proof for $v_{0\alpha}$ is the same by taking α_0 smaller if necessary.

It remains to prove (iii). Here we use Proposition 7.10. We again recall a fundamental property for sup-convolution; if there is an $x_1 \in (\hat{x}, b_1)$ such that $u_0(\hat{t},x_1) < u_0(\hat{t},\hat{x})$, then
$$\sup R(u_0^\alpha(\hat{t},\cdot),\hat{x}) < b_1$$
for sufficiently small α (provided that $u_0(\hat{t},\cdot)$ is upper-semicontinuous which is fulfilled in our setting). Thus Proposition 7.10 implies the desired inclusion by taking smaller α_0 if necessary. $\square$

In Corollary 7.9 we have
$$u_0(t,x) - v_0(s,y) - S(t,s) \leqq u_0(\hat{t},\hat{x}) - v_0(\hat{s},\hat{y}) - S(\hat{t},\hat{s})$$
for all $(x,y) \in \Sigma_\delta,\ t,s \in [0,T]$. We shall derive a similar inequality for u_0^α and $v_{0\alpha}$. We introduce a barrier for $|x - y - (\hat{x} - \hat{y})| > \delta$ so that the region where the inequality is valid contains all $x, y \in \boldsymbol{R}$.

7.13. Proposition. *Let* ϑ *be as in §7.11. Let* $(\hat{t},\hat{x},\hat{s},\hat{y}), u_0$ *and* v_0 *be as in Proposition 7.7.*

(i) $u_0(t,x) - v_0(s,y) - S(t,s) - \vartheta(x-y-\eta,\lambda) \leqq u_0(\hat{t},\hat{x}) - v_0(\hat{s},\hat{y}) - S(\hat{t},\hat{s})$ *for all* $(t,s),(s,y) \in \overline{Q} \times \overline{Q}$ *provided that* $\lambda > 0$ *is sufficiently small, i.e.,* $\lambda \leqq \lambda_0$ *for some* $\lambda_0 > 0$, *where* $\eta = \hat{x} - \hat{y}$.

(ii) *Let* α_0 *be as in Proposition 7.12. Then*
$$u_0^\alpha(t,x) - v_{0\alpha}(s,y) \leqq u_0^\alpha(\hat{t},\hat{x}) - v_{0\alpha}(\hat{s},\hat{y}) + \vartheta(x-y-\eta,\tfrac{1}{2}\lambda_0) + S(t,s) - S(\hat{t},\hat{s})$$
for all $(t,x),(s,y) \in [0,T] \times \boldsymbol{R}$ *provided that* $0 < \alpha \leqq \alpha_1 = \min(\alpha_0, \frac{1}{4}\lambda_0)$.

Proof. (i) Let
$$E_1 = \sup\{E(t,x,s,y);\quad (t,x) \in \overline{Q},\ (s,y) \in \overline{Q},\ (x,y) \in \Sigma_\delta\},$$
$$E_2 = \sup\{E(t,x,s,y);\quad (t,x) \in \overline{Q},\ (s,y) \in \overline{Q},\ (x,y) \notin \Sigma_\delta\}$$

with $E = u_0(t,x) - v_0(s,y) - S(t,s)$ and $\Sigma_\delta = \{(x,y) \in \overline{\Omega} \times \overline{\Omega};\, |x - y - (\hat{x} - \hat{y})| < \delta\}$. Step 1 of Proposition 7.7 yields $E_1 = u_0(\hat{t},\hat{x}) - v_0(\hat{s},\hat{y}) - S(\hat{t},\hat{s})$. If $E_2 \leqq E_1$, the inequality in (i) is trivial since $\vartheta \geqq 0$. If $E_2 > E_1$, we take λ_0 satisfying

$$\vartheta(\delta, \lambda_0) = E_2 - E_1.$$

Since $\vartheta(x - y - \eta, \lambda_0) \geqq \vartheta(\delta, \lambda_0)$ for $(x,y) \notin \Sigma_\delta$ and since ϑ is monotone in λ, the proof of (i) is complete.

(ii) From (i) it follows that

$$\begin{aligned} u_0(t,\tilde{x}) &- \vartheta(x - \tilde{x}, \alpha) - (v_0(s,\tilde{y}) + \vartheta(y - \tilde{y}, \alpha)) \\ &\leqq u_0(\hat{t},\hat{x}) - v_0(\hat{s},\hat{y}) \\ &\quad + \vartheta(\tilde{x} - \tilde{y} - \eta, \lambda_0) - \vartheta(x - \tilde{x}, \alpha) - \vartheta(y - \tilde{y}, \alpha) \\ &\quad + S(t,s) - S(\hat{t},\hat{s}). \end{aligned}$$

Since $u_0 = -v_0 = +\infty$ outside Ω, taking the supremum of both sides for $\tilde{x}, \tilde{y} \in \boldsymbol{R}$ we get

$$\begin{aligned} u_0^\alpha(t,x) - v_{0\alpha}(s,y) \leqq{}& u_0(\hat{t},\hat{x}) - v_0(\hat{s},\hat{y}) \\ &+ \vartheta(x - y - \eta, \lambda_0 - 2\alpha) + S(t,s) - S(\hat{t},\hat{s}) \end{aligned}$$

for all $(t,x), (s,y) \in [0,T] \times \boldsymbol{R}$. Here we have invoked the composition rule for sup-convolution (Lemma 5.5 (ii)):

$$\begin{aligned} \sup \{\, &\vartheta(x - y - \eta, \lambda_0) - \vartheta(x - \tilde{x}, \alpha) - \vartheta(y - \tilde{y}, \alpha); \tilde{x}, \tilde{y} \in \boldsymbol{R}\} \\ &= \vartheta(x - y - \eta, \lambda_0 - 2\alpha). \end{aligned}$$

Since $u_0^\alpha(\hat{t},\hat{x}) = u_0(\hat{t},\hat{x})$ and $v_{0\alpha}(\hat{s},\hat{y}) = v_0(\hat{s},\hat{y})$ by Proposition 7.12(i) and since $\lambda_0 - 2\alpha \leqq \frac{1}{2}\lambda_0$ for $\alpha \leqq \alpha_1$, the proof of (ii) is now complete.

E. Application of the Maximum Principle

The function $u_0^\alpha + p_0 x$ is essentially an admissible superfunction (of the infinitesimal version defined in § 6.5) at $(\hat{t},\hat{x}) \in Q$ except that the faceted region $R(u_0^\alpha(\hat{t},\cdot),\hat{x})$ may contain the boundary point of Ω. We apply the Maximum Principle of Time Direction 4.5 and its Corollary 4.6 to Proposition 7.13 to get useful admissible superfunctions.

Here we use the notation $\tilde{N}^{-1}$ for a semineighborhood defined in §6.3.

7.14. Proposition. *Let $(\hat{t},\hat{x},\hat{s},\hat{y}), u_0$ and v_0 be as in Proposition 7.7. There are a (real-valued) upper-semicontinuous function $\overline{u}$ and a lower-semicontinuous function $\underline{v}$ defined in Q such that*

(i) *$\overline{u}(t,\cdot) \in C(\Omega), \underline{v}(s,\cdot) \in C(\Omega)$ for each $t,s \in (0,T)$; $\overline{u}(\hat{t},\cdot)$ and $\underline{v}(\hat{s},\cdot)$ are faceted at $\hat{x}$ and $\hat{y}$, respectively, with slope zero in Ω; $u_0^\alpha \leqq \overline{u}$ and $v_{0\alpha} \geqq \underline{v}$ in Q; $\overline{u}(\hat{t},\hat{x}) = u_0^\alpha(\hat{t},\hat{x})$ and $\underline{v}(\hat{s},\hat{y}) = v_{0\alpha}(\hat{s},\hat{y})$.*

(ii) $R(\overline{u}(\hat{t},\cdot),\hat{x}) = R(u_0^\alpha(\hat{t},\cdot),\hat{x}) \cap \{R(v_{0\alpha}(\hat{s},\cdot),\hat{y}) + \eta\}$,

$$R(\underline{v}(\hat{s},\cdot),\hat{y}) = \{R(u_0^\alpha(\hat{t},\cdot),\hat{x}) - \eta\} \cap R(v_{0\alpha}(\hat{s},\cdot),\hat{y}),$$

so that $L(\overline{u}(\hat{t},\cdot),\hat{x}) = L(\underline{v}(\hat{s},\cdot),\hat{y})$, *where* $\eta = \hat{x} - \hat{y}$.

(iii) $\chi(\overline{u}(\hat{t},\cdot),\hat{x}) + \chi(-\underline{v}(\hat{s},\cdot),\hat{y}) \leqq 0$.

(iv) *For some* $\tilde{N}^{-1}(\overline{u}(\hat{t},\cdot),\hat{x})$ *the inequality*

$$\overline{u}(t,x) - \overline{u}(\hat{t},\hat{x}) \leqq S(t,\hat{s}) - S(\hat{t},\hat{s})$$

holds for all $(t,x) \in (0,T) \times \tilde{N}^{-1}(\overline{u}(\hat{t},\cdot),\hat{x})$; *for some* $\tilde{N}^{-1}((-\overline{v})(\hat{s},\cdot),\hat{y})$ *the inequality*

$$\underline{v}(s,y) - \underline{v}(\hat{s},\hat{y}) \geqq S(\hat{t},\hat{s}) - S(\hat{t},s)$$

holds for all $(s,y) \in (0,T) \times \tilde{N}^{-1}(-\underline{v}(\hat{s},\cdot),\hat{y})$. *Here* α *is chosen so that* $\alpha \leqq \alpha_1$, *where* α_1 *is as in Proposition 7.13.*

Proof. We apply Corollary 4.6 of the Maximum Principle to Propositions 7.12 and 7.13 so that we find the desired $\overline{u}$ and $\underline{v}$. Here u_0^α, and $-v_{0\alpha}$ correspond to u_1, and u_2 of Corollary 4.6, respectively. Note that Proposition 7.12(iii) plays an important role in applying Corollary 4.6.

7.15. Proposition. *Let*

$$U(t,x) = \overline{u}(t,x) + p_0 x, \quad V(s,y) = \underline{v}(s,y) + p_0(y + \zeta_0) \quad \textit{for } (t,x),(s,y) \in Q.$$

Then U *is an admissible superfunction at* $(\hat{t},\hat{x})$ *in* Q, *and* V *is an admissible subfunction at* $(\hat{t},\hat{x})$ *in* Q. *Moreover,* $U(\hat{t},\cdot)$ *is faceted at* $\hat{x} \in \operatorname{int} R(U(\hat{t},\cdot),\hat{x})$ *and* $\mathscr{T}_P^{+}U(\hat{t},\hat{x}) \ni S_t(\hat{t},\hat{s})$; $V(\hat{s},\cdot)$ *is faceted at* $\hat{y} \in \operatorname{int} R(V(\hat{s},\cdot),\hat{y})$ *and* $\mathscr{T}_P^{-}V(\hat{s},\hat{y}) \ni -S_s(\hat{t},\hat{s})$, *where* $\mathscr{T}_P^{+}$ *and* $\mathscr{T}_P^{-}$ *are as in* §*6.4.*

Proof. We only prove that U is an admissible superfunction since the statement for V can be proved similarly. By Proposition 7.14(i) U is upper-semicontinuous in Q and $U(t,\cdot) \in C(\Omega)$ for each $t \in (0,T)$. Moreover, $U(\hat{t},\cdot)$ is P-faceted at $\hat{x}$ in Ω with slope p_0. Since $\hat{x}$ and $\hat{y}$ are respectively interior points of $R(u_0^\alpha(\hat{t},\cdot),\hat{x})$ and $R(v_{0\alpha}(\hat{s},\cdot),\hat{y})$ by Proposition 7.12(ii), we see, by Proposition 7.14(ii) that $\hat{x}$ and $\hat{y}$ are resepectively interior points of $R(U(\hat{t},\cdot),\hat{x})$ and $R(V(\hat{s},\cdot),\hat{y})$.

It remains to prove that $\mathscr{T}_P^{+}U(\hat{t},\hat{x})$ contains $S_t(\hat{t},\hat{s})$ as an element. Since

$$S(t,\hat{s}) - S(\hat{t},\hat{s}) \leqq S_t(\hat{t},\hat{s})(t-\hat{t}) + \omega(|t-\hat{t}|)|t-\hat{t}|$$

for all $t \in \boldsymbol{R}$ with some modulus ω, Proposition 7.14(iv) yields

$$U(t,x) - U(\hat{t},\hat{x}) \leqq p_0(x-\hat{x}) + S_t(\hat{t},\hat{s})(t-\hat{t}) + \omega(|t-\hat{t}|)|t-\hat{t}|$$

for $(t,x) \in (0,T) \times \tilde{N}^{-1}(U(\hat{t},\cdot),\hat{x})$. This implies that $S_t(\hat{t},\hat{s}) \in \mathscr{T}_P^{+}U(\hat{t},\hat{x})$.

F. Completion of the Proof for Case I

By definition of sub- and supersolutions u^* and $-v_*$ are upper-semicontinuous in $[0, T) \times \overline{\Omega}$ with values in $\boldsymbol{R} \cup \{-\infty\}$. To prove our Comparison Theorem, it suffices to prove that the restrictions of u^* and v_* to $[0, T') \times \overline{\Omega}$ (denoted by u' and v' respectively) satisfy $u' \leqq v'$ on $[0, T') \times \overline{\Omega}$ for each $T' > 0$. Clearly $(u')^* - (v')_* \leqq u^* - v_*$ on $[0, T'] \times \overline{\Omega}$, so that $u^* \leqq v_*$ on $\partial_p Q$ implies that $(u')^* \leqq (v')_*$ on

$$\overline{\partial}_p Q' = \overline{\partial_p Q'} = [0, T'] \times \partial\Omega \cup \{0\} \times \overline{\Omega}.$$

By replacing T' by T, u' by u, and v' by v, we may assume that $u \leqq v$ on $\overline{\partial}_p Q$ and that u and $-v$ are upper-semicontinuous in $\overline{Q}$ with values in $\boldsymbol{R} \cup \{-\infty\}$. If F depends on t, by replacing T' by T we may assume that

(F3′) F is uniformly continuous on $[0, T] \times [-K, K] \times \boldsymbol{R}$ for each $K > 0$.

Suppose that the conclusion of the theorem were false. We may assume that m_0 satisfies the assumption of Proposition 7.1. We fix $\varepsilon_0, \sigma_0, \gamma_0, \gamma_0'$ as in Remark 7.2 and assume that $0 < \varepsilon < \varepsilon_0,\ 0 < \sigma < \sigma_0,\ 0 < \gamma < \gamma_0,\ 0 < \gamma' < \gamma_0'$. Since $\overline{Q}$ is compact and u and $-v$ are upper-semicontinuous, there is always a maximizer $(t_\zeta, x_\zeta, s_\zeta, y_\zeta)$ of Φ_ζ over $\overline{Q} \times \overline{Q}$ in Proposition 7.1.

If we assume Case I, we end up with Propositions 7.12–7.15. Let p_0 and ζ_0 be as in Lemma 7.4. Let $(\hat{t}, \hat{x}, \hat{s}, \hat{y})$ be a maximizer of Φ_{ζ_0} with $p_0 = 2(\hat{x} - \hat{y} - \zeta_0)/\varepsilon$. Let U be as in Proposition 7.15. Then

$$\max_Q(u - U) = \max_Q(u_0 - \overline{u}) \leqq \max_Q(u_0 - u_0^\alpha)$$

since $u_0^\alpha \leqq \overline{u}$ by Proposition 7.14(i). Note that $u_0^\alpha(\hat{t}, \hat{x}) = u_0(\hat{t}, \hat{x})$ by Proposition 7.12(i) and that $u_0 \leqq u_0^\alpha$ by the definition of sup-convolution. We thus observe that

$$\max_Q(u - U) = 0.$$

Similarly, one can prove

$$\min_Q(v - V) = 0.$$

Since U is an admissible superfunction at $(\hat{t}, \hat{x})$ in Q and since u is a subsolution we have, by the definition of the infinitesimal version 6.6, and Theorems 6.8 and 6.9,

$$S_t(\hat{t}, \hat{s}) + F(\hat{t}, p_0, \Delta\chi_U/L_U) \leqq 0$$

with $\chi_U = \chi(U(\hat{t}, \cdot), \hat{x}), L_U = L(U(\hat{t}, \cdot), \hat{x})$ and $\Delta = W'(p_0 + 0) - W'(p_0 - 0)$. Similarly,

$$-S_s(\hat{t}, \hat{s}) + F(\hat{s}, p_0, \Delta\chi_V/L_V) \geqq 0.$$

with $\chi_V = \chi(V(\hat{s}, \cdot), \hat{y}), L_V = L(V(\hat{s}, \cdot), \hat{y})$. Subtracting the second inequality from the first yields

$$\frac{\gamma}{(T - \hat{t})^2} + \frac{\gamma'}{(T - \hat{s})^2} + F(\hat{t}, p_0, \Delta\chi_U/L_U) - F(\hat{s}, p_0, \Delta\chi_V/L_V) \leqq 0.$$

By Proposition 7.14(ii), (iii) we see that

$$\begin{aligned}\chi_U &= \chi(\overline{u}(\hat{t},\cdot),\hat{x}) \leqq -\chi(-\underline{v}(\hat{s},\cdot),\hat{y}) = \chi(\underline{v}(\hat{s},\cdot),\hat{y}) = \chi_V\,,\\ L_U &= L(\overline{u}(\hat{t},\cdot),\hat{x}) = L(\underline{v}(\hat{s},\cdot),\hat{y}) = L_V\,,\end{aligned}$$

which yield

$$\frac{\chi_U}{L_U} \leqq \frac{\chi_V}{L_V}.$$

By the monotonicity (F2) of F we now obtain

$$\text{(7.1)}\qquad \frac{\gamma}{(T-\hat{t})^2} + \frac{\gamma'}{(T-\hat{s})^2} + F\left(\hat{t}, p_0, \Delta\frac{\chi_U}{L_U}\right) - F\left(\hat{s}, p_0, \Delta\frac{\chi_U}{L_U}\right) \leqq 0.$$

This yields

$$\frac{\gamma+\gamma'}{T^2} \leqq \omega_1(|\hat{t}-\hat{s}|;\varepsilon),$$

where $\omega_1(t;\varepsilon)$ is a modulus of continuity of $F(t,p,X)$ for $|p| \leqq K$ and $X \in \boldsymbol{R}$ with $K \equiv 2M^{1/2}/\varepsilon^{1/2}$, provided that $|p_0| \leqq K$; the existence of such a modulus is guaranteed by (F3′). Indeed, since Proposition 7.1(ii) implies that

$$|B_\varepsilon'(x_\zeta - y_\zeta - \zeta)| = |2(x_\zeta - y_\zeta - \zeta)/\varepsilon| \leqq 2M^{1/2}/\varepsilon^{1/2} = K \quad \text{for } \zeta, |\zeta| \leqq \kappa_0(\varepsilon),$$

we have $|p_0| \leqq K$. Note that K depends only on ε and is independent of parameters σ, γ, γ'. Thus ω_1 depends only on ε and is independent of parameters σ, γ, γ'. By Proposition 7.1(ii) we have

$$\frac{\gamma+\gamma'}{T^2} \leqq \omega_1(|\hat{t}-\hat{s}|;\varepsilon) \leqq \omega_1((M\sigma)^{1/2};\varepsilon)$$

for all $\varepsilon, \sigma, \gamma, \gamma'$ with $0 < \varepsilon < \varepsilon_0, 0 < \sigma < \sigma_0, 0 < \gamma < \gamma_0, 0 < \gamma' < \gamma_0'$ provided that Case I holds. As shown in Section G, a similar inequality (7.3) holds for Case II with modulus $\omega_2(\cdot,\varepsilon)$ independent of σ, γ, γ'. In both cases if σ is taken so small that

$$\omega_i((M\sigma)^{1/2}) < \frac{\gamma+\gamma'}{T^2}, \quad i = 1, 2,$$

then we get a contradiction. If F is independent of t, (7.1) immediately yields a contradiction: $(\gamma+\gamma')/T^2 < 0$ without using (F3′).

G. Case II

We use the same choice of parameters $\varepsilon, \gamma, \gamma'$ and σ as in § F. The proof for Case II is standard so we just outline it. By the assumption of Case II there is $\zeta_j \to 0$ such that $\hat{\Psi}_x = 2(x_j - y_j - \zeta_j)/\varepsilon$ does not belong to P for some maximizer (t_j, x_j, s_j, y_j) of Φ_{ζ_j}. We apply the maximum principle for semicontinuous functions [CIL] and observe that for each $\mu > 0$ there are 2×2 symmetric matrices X and Y such that

$$(\hat{\Psi}_t, \hat{\Psi}_x, X) \in \overline{J}_Q^{2,+} u(t_j, x_j),$$

$$(\hat{\Psi}_s, \hat{\Psi}_y, -Y) \in \overline{J}_Q^{2,+}(-v)(s_j, y_j),$$

$$-\left(\frac{1}{\mu} + |A|\right) I \leqq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqq A + \mu A^2$$

with $A = D^2\Psi_{\zeta_j}(t_j, x_j, s_j, y_j)$ i.e., the Hessian of Ψ_{ζ_j}, where $|A|$ is the operator norm of A as a self-adjoint operator and I is a 4×4 matrix; $\hat{\Psi}_t$ and $\hat{\Psi}_y$ are the derivatives of Ψ_{ζ_j} with respect to t and y at (t_j, x_j, s_j, y_j), respectively. Here $J_Q^{2,+}$ denotes the set of second-order superjets and $\overline{J}_Q^{2,+}$ is the 'closure' in the sense of semijets; see [CIL] for definitions. It is not difficult [OKS] to derive from the first two inclusions that

$$(\hat{\Psi}_t, \hat{\Psi}_x, X_{22}) \in \overline{\mathscr{P}}_Q^{2,+} u(t_j, x_j),$$

$$(-\hat{\Psi}_s, -\hat{\Psi}_y, Y_{22}) \in \overline{\mathscr{P}}_Q^{2,-} v(s_j, y_j)$$

where $\overline{\mathscr{P}}_Q^{2,\pm}$ denotes the parabolic version of $\overline{J}_Q^{2,\pm}$; $\overline{\mathscr{P}}_Q^{2,\pm}$ is defined in §6.6. Here X_{22} and Y_{22} denote the (2, 2)-components of X and Y respectively. By a standard argument, our matrix inequality yields $X_{22} \leqq Y_{22}$.

Since $\hat{\Psi}_x = -\hat{\Psi}_y$ does not belong to P and since u and v are sub- and supersolutions (by Definition 6.6 and Theorems 6.8 and 6.9), we have

$$\hat{\Psi}_t + F(t_j, \hat{\Psi}_x, W''(\hat{\Psi}_x)X_{22}) \leqq 0,$$

$$-\hat{\Psi}_s + F(s_j, -\hat{\Psi}_y, W''(-\hat{\Psi}_y)Y_{22}) \geqq 0,$$

where X_{22} and Y_{22} are the $(2, 2)$-components of X and Y. Subtracting the second from the first of these inequalities yields

$$\begin{aligned} \frac{\gamma}{(T - t_j)^2} + \frac{\gamma'}{(T - s_j)^2} &+ F(t_j, \hat{\Psi}_x, W''(\hat{\Psi}_x)X_{22}) \\ &- F(s_j, \hat{\Psi}_x, W''(\hat{\Psi}_x)X_{22}) \leqq 0 \end{aligned} \tag{7.2}$$

if we use $X_{22} \leqq Y_{22}$, (F2) and $\hat{\Psi}_x = -\hat{\Psi}_y$. Since Proposition 7.1(ii) yields that $|\hat{\Psi}_x|$ is bounded by K in § F, as in § F this inequality yields

$$\frac{\gamma + \gamma'}{T^2} \leqq \omega_2(|t_j - s_j|; \varepsilon) \leqq \omega_2((M\sigma)^{1/2}; \varepsilon) \tag{7.3}$$

with some modulus ω_2 for all $\varepsilon, \sigma, \gamma, \gamma'$ with $0 < \varepsilon < \varepsilon_0$, $0 < \sigma < \sigma_0$, $0 < \gamma < \gamma_0$, $0 < \gamma' < \gamma'_0$, provided that Case II holds. This is what we would like to prove. (We did not send $\zeta_j \to 0$; the existence of one ζ_j is enough to get a contradiction.) Note that (7.2) immediately yields a contradiction $(\gamma + \gamma')/T^2 < 0$ if F is independent of t; we do not invoke (F3′). We have thus proved our Comparison Theorem.

H. Periodic Case, Proof of Theorem 3.2

The method of the proof is essentially the same. Since u and v are ϖ-periodic in space, we observe that any maximizer $(t_\zeta, x_\zeta, s_\zeta, y_\zeta)$ of Φ_ζ satisfies $|x_\zeta - y_\zeta| \leqq \varpi$; we may always assume its existence and that $0 \leqq x_\zeta \leqq \varpi$, $0 \leqq y_\zeta \leqq \varpi$. The conclusion of Proposition 7.1 and Remark 7.2 are still valid for such maximizers where Ω is replaced by $\boldsymbol{R}$. To define $\mathscr{A}(\zeta)$ we replace Ω by $\boldsymbol{R}$. Although graph $\mathscr{A}$ is only closed, graph $\mathscr{B}$ is still compact as claimed in Proposition 7.3 since $0 \leqq x_\zeta, y_\zeta \leqq \varpi$. The argument in §§7.4–7.9 is still valid if Ω is replaced by $\boldsymbol{R}$. Propositions 7.10 and 7.12(iii) should be altered because there is no boundary of $\boldsymbol{R}$. If P does not contain zero so that $p_0 \neq 0$, then the faceted regions $R(u_0^\alpha(\hat{t}, \cdot), \hat{x})$ and $R(v_{0\alpha}(\hat{s}, \cdot), \hat{y})$ have length less than ϖ since $u_0^\alpha + p_0 x$ and $v_{0\alpha} + p_0(y + \zeta_0)$ are periodic in the space variable. We apply the maximum principle as in Proposition 7.14. Actually, we have Proposition 7.14 with Ω replaced by

$$\Omega' = (\hat{x} - \varpi, \hat{x} + \varpi) \cup (\hat{y} - \varpi, \hat{y} + \varpi),$$

since Proposition 7.13 does not apply to our setting with $\Omega = \boldsymbol{R}$; the rest of the proof is the same as in §§E–G.

It remains to discuss the case that P contains zero. If $p_0 \neq 0$ in Lemma 7.4, then we argue in the same way. If $p_0 = 0$, then either $L(u_0^\alpha(\hat{t}, \cdot), \hat{x}) < \varpi$ or $R(u_0^\alpha(\hat{t}, \cdot), \hat{x}) = \boldsymbol{R}$ and the same holds for $v_{0\alpha}(\hat{s}, \cdot)$. Unless both

$$R(u_0^\alpha(\hat{t}, \cdot), \hat{x}) = \boldsymbol{R} \text{ and } R(v_{0_\alpha}(\hat{s}, \cdot), \hat{y}) = \boldsymbol{R}$$

hold, Proposition 7.14 still applies with Ω replaced by Ω'. We thus assume that both $u_0^\alpha(\hat{t}, \cdot)$ and $v_{0\alpha}(\hat{s}, \cdot)$ are constant functions. For each large l we consider a nonnegative continuous function f such that $f(x) = 0$ if and only if $|x| \leqq \frac{1}{2} l$. We set $\bar{u}(t, x) = u_0^\alpha(t, x) + f(x - \hat{x}), \underline{v}(s, y) = v_{0\alpha}(s, y) - f(y - \hat{y})$ and observe that properties (i), (iv) of Proposition 7.14 (with $\Omega = \boldsymbol{R}$) hold; note that since $\chi(\bar{u}(\hat{t}, \cdot), \hat{x}) > 0$ and $\chi(-\underline{v}(\hat{s}, \cdot), \hat{y}) > 0$, it follows that (iii) is violated but $\tilde{N}^{-1}(\bar{u}(\hat{t}, \cdot), \hat{x}) = R(\bar{u}(\hat{t}, \cdot), \hat{x})$ and $\tilde{N}^{-1}(-\underline{v}(\hat{s}, \cdot), \hat{y}) = R(\underline{v}(\hat{s}, \cdot), \hat{y})$ so (iv) trivially follows from Proposition 7.13. We also note that $L(\bar{u}(\hat{t}, \cdot), \hat{x}) = L(\underline{v}(\hat{s}, \cdot), \hat{y}) = l$. We may apply Proposition 7.15 with $U = \bar{u}$ and $V = \underline{v}$. As in §F we end up with

$$\frac{\gamma}{(T - \hat{t})^2} + \frac{\gamma'}{(T - \hat{s})^2} + F(\hat{t}, 0, \Delta\chi_U / L_U) - F(\hat{s}, 0, \Delta\chi_V / L_V) \leqq 0$$

which yields

$$\frac{\gamma + \gamma'}{T^2} \leqq \omega_3(|2\Delta / l| + |\hat{t} - \hat{s}|) \quad \text{for large } l$$

since $L_U = L_V = l$. Here ω_3 is a modulus of continuity of $F(t, 0, X)$ on $[0, T] \times [-1, 1]$; here we do not invoke (F3′). Sending $l \to \infty$ yields

$$\frac{\gamma + \gamma'}{T^2} \leqq \omega_3(|\hat{t} - \hat{s}|),$$

which again leads to a contradiction as in §F.

Our argument shows at least formally that the weighted curvature

$$\Lambda_W(u_0^\alpha(\hat{t}, \cdot), \hat{x}) = 0 \quad \text{if } L(u_0^\alpha(\hat{t}, \cdot), \hat{x}) = \infty.$$

8. Perron-Type Existence Theorem

We give the proof of the Perron-type Existence Theorem 3.3 and Theorem 3.4 for periodic functions. Let Ω be an open (possibly unbounded) interval and $Q = (0,T) \times \Omega$.

8.1. Lemma. *Assume that conditions* (F1) *and* (F2) *hold. Let S be a nonempty family of subsolutions of* (E). *Let u be a function defined on Q by*

$$u(t,x) = \sup\{v(t,x); v \in S\} \quad \textit{for } (t,x) \in Q.$$

Suppose that $u^* < \infty$ in $[0,T) \times \overline{\Omega}$. Then u is a subsolution of (E).

8.1′. Lemma. *Assume that conditions* (F1) *and* (F2) *hold. Let S be a nonempty family of supersolutions of* (E). *Let u be a function defined on Q by*

$$u(t,x) = \inf\{v(t,x); v \in S\} \quad \textit{for } (t,x) \in Q.$$

Suppose that $u_* > -\infty$ in $[0,T) \times \overline{\Omega}$. Then u is a supersolution of (E).

8.2. Lemma. *Assume that conditions* (F1) *and* (F2) *hold. Let $h : Q \to \boldsymbol{R}$ be a supersolution of* (E). *Let S be the collection of all subsolutions v of* (E) *with $v \leqq h$ in Q. If $v \in S$ is not a supersolution of (E) and $v_* > -\infty$ in $[0,T) \times \overline{\Omega}$, then there are a function $w \in S$ and a point $(s,y) \in Q$ such that $v(s,y) < w(s,y)$.*

Theorem 3.3 follows from these two lemmas as in [I]. We give its proof for completeness.

Proof of Perron-Type Existence Theorems 3.3 and 3.4 under the Assumption That Lemmas 8.1 and 8.2 Hold. Let S be $\{v;\ v$ is a subsolution of (E) and $v \leqq u^+$ in $Q\}$. Since $u^- \in S$, we see that $S \neq \emptyset$. Let $u : Q \to \boldsymbol{R}$ be defined by

$$u(t,x) = \sup\{v(t,x);\ v \in S\} \quad \text{for } (t,x) \in Q.$$

By definition, $u^* \leqq (u^+)^* < \infty$ in $[0,T) \times \overline{\Omega}$. By Lemma 8.1, u is a subsolution of (E), so that $u \in S$. Since $u^- \in S$, we have $u^- \leqq u \leqq u^+$ in Q and $-\infty < (u^-)_* \leqq u_*$ in $[0,T) \times \overline{\Omega}$.

Suppose that u were not a supersolution of (E). By Lemma 8.2 there would exist $w \in S$ and $(s,y) \in Q$ such that $u(s,y) < w(s,y)$. This contradicts the definition of u. Therefore u is a generalized solution of (E).

Suppose that u were not ϖ -periodic in x. Then there would exist a point $(t_0,x_0) \in (0,T) \times \boldsymbol{R}$ and $\nu \in \{-1,1\}$ that satisfy

$$u(t_0,x_0) < u(t_0,x_0 + \nu\varpi).$$

We observe that

$$v(t,x) = u(t,x + \nu\varpi)$$

belongs to S. Indeed, since (E) is invariant under translation in x, v is a subsolution of (E) in $(0,T) \times \boldsymbol{R}$. The function u^+ is ϖ -periodic, so $v \leqq u^+$ in $(0,T) \times \boldsymbol{R}$, which yields $v \in S$. By the definition of u, the property $v \in S$ implies that $u \geqq v$. This contradicts

$$u(t_0, x_0) < v(t_0, x_0) = u(t_0, x_0 + \nu\varpi).$$

We thus conclude u is ϖ-periodic in x. $\square$

To prove Lemmas 8.1 and 8.2, we extend the method found in [I] to faceted functions. The basic strategy is to utilize the advantage of our definition of C_P^2 functions. We have arranged that the weighted curvature of a C_P^2 function at the boundary of a faceted region *equals* that at an interior point of the faceted region. Moreover, we have arranged that the weighted curvature of a C_P^2 function at point y_k outside the faceted region tends to zero, if the point y_k tends to a point $\bar{y}$ of the boundary of the faceted region as $k \to +\infty$, since the function is C^2. These properties will be invoked to estimate the weighted curvature of a modification of a test function, which we call an upper or a lower canonical modification as defined below.

8.3. Notation of An Upper and a Lower Canonical Modification. Let Ω_1 be an open interval with $\Omega_1 \subset \Omega$. Let $f \in C(\Omega)$ satisfy $f|_{\Omega_1} \in C_P^2(\Omega_1)$ and $f'(\hat{x}) = 0$ with $\hat{x} \in \Omega_1$. Let $q_1 = \sup\{p \in P \cup \{-\infty\}; p < 0\}$ and $q_2 = \inf\{p \in P \cup \{+\infty\}; p > 0\}$.

Case(i) ($0 \notin P$). Set

$$f^{\#}(x) = f(x) + (x - \hat{x})^4 \quad \text{for } x \in \Omega.$$

There exists an open interval $\Omega_2 \subset \Omega_1$ containing $\hat{x}$ such that

$$\tfrac{1}{2}q_1 < (f^{\#})'(x) < \tfrac{1}{2}q_2 \quad \text{for all } x \in \Omega_2. \tag{8.1}$$

Case(ii) ($0 \in P$). We see that f is P-faceted at $\hat{x}$ in Ω_1. We denote $\Omega = (a_-, a_+)$ and $R(f, \hat{x}) = [c_-, c_+]$. There exists an open interval $\Omega_2 = (b_-, b_+)$ such that $R(f, \hat{x}) \subset \Omega_2 \subset \Omega_1$ and

$$f'(x) \in (\tfrac{1}{4}q_1, 0) \cap (0, \tfrac{1}{4}q_2) \quad \text{for all } x \in \Omega_2 \backslash R(f, \hat{x}),$$

$$b_+ \leqq c_+ + (\tfrac{1}{16}q_2)^{1/3}, \quad b_- \geqq c_- - (-\tfrac{1}{16}q_1)^{1/3}.$$

If $\chi_+(f, \hat{x}) = 1$, then we set

$$f^{\#}(x) = \begin{cases} f(\hat{x}) & \text{for } x \in [\hat{x}, c_+], \\ f(x) + (x - c_+)^4 & \text{for } x \in (c_+, a_+). \end{cases}$$

If $\chi_-(f, \hat{x}) = 1$, then we set

$$f^{\#}(x) = \begin{cases} f(\hat{x}) & \text{for } x \in [c_-, \hat{x}), \\ f(x) + (x - c_-)^4 & \text{for } x \in (a_-, c_-). \end{cases}$$

If $\chi_+(f, \hat{x}) = -1$, then for $\varepsilon \in (0, \varepsilon_0)$ we set

$$f^{\#,\varepsilon}(x)=\begin{cases} f(\hat{x}) & \text{for } x\in[\hat{x},c_+ +\varepsilon],\\ f(x-\varepsilon) & \text{for } x\in(c_+ +\varepsilon,b_+),\\ f(x)+\{f(b_+ -\varepsilon)-f(b_+)\} & \text{for } x\in(b_+,a_+).\end{cases}$$

If $\chi_-(f,\hat{x})=-1$, then for $\varepsilon\in(0,\varepsilon_0)$ we set

$$f^{\#,\varepsilon}(x)=\begin{cases} f(\hat{x}) & \text{for } x\in[c_- -\varepsilon,\hat{x}),\\ f(x+\varepsilon) & \text{for } x\in(b_-,c_- -\varepsilon),\\ f(x)+\{f(b_- +\varepsilon)-f(b_-)\} & \text{for } x\in(a_-,b_-].\end{cases}$$

where $\varepsilon_0=\frac{1}{2}\operatorname{dist}(\partial\Omega_2,R(f,\hat{x}))$. Here we note that (8.1) holds.

We often suppress the ε-dependence of $f^{\#,\varepsilon}$, even if it depends on ε. We call $f^{\#}$ *an upper canonical modification of* f *at* $\hat{x}$ *with canonical neighborhood* Ω_2. Let $-f_{\#}$ be an upper canonical modification of $-f$ at $\hat{x}$ with canonical neighborhood Ω_2; we call $f_{\#}$ *a lower canonical modification of* f *at* $\hat{x}$ *with canonical neighborhood* Ω_2.

8.4. Proposition. *Let* Ω *and* Ω_1 *be open intervals with* $\Omega_1\subset\Omega$. *Let* $f\in C(\Omega)$ *satisfy* $f|_{\Omega_1}\in C_P^2(\Omega_1)$ *and* $f'(\hat{x})=0$ *with* $\hat{x}\in\Omega_1$. *Suppose that* $\tilde{f}$ *is an upper canonical modification* $f^{\#}$ $(=f^{\#,\varepsilon})$ *or a lower canonical modification* $f_{\#}$ $(=f_{\#,\varepsilon})$ *at* $\hat{x}$ *with canonical neighborhood* $\Omega_2\subset\Omega_1$. *Set*

$$s=\begin{cases} 1 & \textit{if } \tilde{f} \textit{ is an upper canonical modification,}\\ -1 & \textit{if } \tilde{f} \textit{ is a lower canonical modification,}\end{cases}$$

$$M=\begin{cases}\{\hat{x}\} & \textit{if } 0\notin P,\\ R(f,\hat{x}) & \textit{if } 0\in P.\end{cases} \tag{8.2}$$

(i) *Then*

$$\tilde{f}\in C(\Omega),\quad \tilde{f}|_{\Omega_2}\in C_P^2(\Omega_2),$$
$$(\tilde{f})^{(n)}(x)=f^{(n)}(\hat{x})\quad \textit{for } x\in M \textit{ and } n=0,1,2,$$
$$s\tilde{f}>sf\quad \textit{in } \Omega\backslash M,$$

$$\begin{aligned}\Lambda_W(\tilde{f},\hat{x}) &= \Lambda_W(f,\hat{x}) && \textit{if } s\chi(f,\hat{x})\geqq 0,\\ s\Lambda_W(\tilde{f},\hat{x}) &> s\Lambda_W(f,\hat{x}) && \textit{if } s\chi(f,\hat{x})=-1.\end{aligned}$$

(ii) *When* $0\in P$,

$$\tilde{f} \textit{ is } P\textit{-faceted at } \hat{x} \textit{ in } \Omega_2 \textit{ with slope } 0,$$
$$\chi_+(\tilde{f},\hat{x})=\chi_+(f,\hat{x}),\ \chi_-(\tilde{f},\hat{x})=\chi_-(f,\hat{x}),$$
$$L(\tilde{f},\hat{x})=L(f,\hat{x})+\{1-s\chi(f,\hat{x})\}\varepsilon.$$

(iii) *For any* $\varepsilon_1>0$ *and* $\varepsilon_2>0$, *there exists an open interval* Ω_3 *such that*

$$M\subset\Omega_3\subset\Omega_2,$$

$$|(\tilde{f})'(x)| < \varepsilon_1 \quad \textit{for } x \in \Omega_3,$$
$$|\Lambda_W(\tilde{f}, x) - \Lambda_W(f, \hat{x})| < \varepsilon_2 \quad \textit{for } x \in \Omega_3.$$

Proof of Lemma 8.1. Step 1. Let $(\hat{t}, \hat{x}) \in Q$. Let $\varphi \in C(\Omega)$ be locally admissible at $(\hat{t}, \hat{x})$ in Q. Suppose that

$$\max_Q (u^* - \varphi) = (u^* - \varphi)(\hat{t}, \hat{x}).$$

Our goal is to prove that

$$\varphi_t(\hat{t}, \hat{x}) + F(\hat{t}, \varphi_x(\hat{t}, \hat{x}), \Lambda_W(\varphi(\hat{t}, \cdot), \hat{x})) \leqq 0 \tag{8.3}$$

(cf. Theorem 6.8). Without loss of generality we may assume that $(u^* - \varphi)(\hat{t}, \hat{x}) = 0$, since $\varphi(t, x)$ can be replaced by $\varphi(t, x) + (u^* - \varphi)(\hat{t}, \hat{x})$. We may assume that $\varphi_x(\hat{t}, \hat{x}) = 0$ by Proposition 2.7 with $A = \varphi_x(\hat{t}, \hat{x})$ and $B = -\varphi_x(\hat{t}, \hat{x})\hat{x}$.

Since φ is locally admissible at $(\hat{t}, \hat{x})$ in Q, there exists a rectangular neighborhood $Q_1 = I \times \Omega_1$ at $(\hat{t}, \hat{x})$ in Q such that $\varphi|_{Q_1} \in A_P(Q_1)$. So there exist $f \in C_P^2(\Omega_1)$ and $g \in C^1(I)$ such that

$$\varphi(t, x) = f(x) + g(t) \quad \text{for } (t, x) \in Q_1.$$

The inequality (8.3) becomes

$$g'(\hat{t}) + F(\hat{t}, 0, \Lambda_W(f, \hat{x})) \leqq 0, \tag{8.4}$$

which we should prove.

Let ζ be a function on Q satisfying

$$\begin{gathered} \zeta \in C(Q), \quad \zeta(\hat{t}, \hat{x}) = 0, \quad \zeta \geqq 0 \text{ in } Q, \\ \{(t, x) \in Q; \zeta(t, x) = 0\} \cap \bar{\partial}_p Q = \emptyset, \end{gathered} \tag{8.5}$$

where $\bar{\partial}_p Q = [0, T] \times \partial\Omega \cap \{0\} \times \bar{\Omega}$. The function ζ is to be determined later. Setting $\psi = \varphi + \zeta$ in Q, we see that

$$\max_Q (u^* - \psi) = (u^* - \psi)(\hat{t}, \hat{x}),$$

so that

$$(u^* - \psi)(t, x) + \zeta(t, x) = (u^* - \varphi)(t, x) \leqq (u^* - \varphi)(\hat{t}, \hat{x}) = 0 \quad \text{for } (t, x) \in Q,$$

which implies that

$$(u^* - \psi) \leqq -\zeta \quad \text{in } Q.$$

By the definition of the upper-semicontinuous envelope, there exists a sequence $\{(t_k, x_k)\}_{k=1}^{\infty} \subset Q$ such that $(t_k, x_k) \to (\hat{t}, \hat{x})$ as $k \to \infty$ and

$$\lim_{k \to \infty} (u^* - \psi)(t_k, x_k) = (u^* - \psi)(\hat{t}, \hat{x}) = 0.$$

By the definition of u, there is a sequence $\{v_k\}_{k=1}^{\infty} \subset \underline{S}$ such that $v_k(t_k, x_k) > u(t_k, x_k) - 1/k$. So there is a sequence $\{(s_k, y_k)\}_{k=1}^{\infty} \subset \overline{Q}$ such that

$$\max_{\overline{Q}}(v_k^* - \psi) = (v_k^* - \psi)(s_k, y_k).$$

These yield

$$\begin{aligned} u^*(t_k, x_k) - 1/k - \psi(t_k, x_k) &< (v_k^* - \psi)(t_k, x_k) \\ &\leqq (v_k^* - \psi)(s_k, y_k) \leqq (u^* - \psi)(s_k, y_k) \leqq -\zeta(s_k, y_k), \end{aligned}$$

so that $\lim_{k\to\infty} \zeta(s_k, y_k) = 0$ since the first term of the last inequality converges to 0 as k tends to $+\infty$. So we get $\lim_{k\to\infty}(u^* - \psi)(s_k, y_k) = 0$ and $(\overline{s}, \overline{y}) \in \{(t, x) \in Q; \zeta(t, x) = 0\}$, where $(\overline{s}, \overline{y}) = \lim_{k\to\infty}(s_k, y_k)$ by taking a subsequence. Since the zero set of ζ does not intersect $\partial_P Q$, for sufficiently large $k > 0$ we have $(s_k, y_k) \in Q$ and

$$\max_{Q}(v_k^* - \psi) = (v_k^* - \psi)(s_k, y_k).$$

Since v_k is a subsolution of (E), we have

$$\psi_t(s_k, y_k) + F(s_k, \psi_x(s_k, y_k), \Lambda_W(\psi(s_k, \cdot), y_k)) \leqq 0 \tag{8.6}$$

provided that ψ is locally admissible near (s_k, y_k) (Proposition 6.8).

Step 2. Let $f^{\#} \in C(\Omega)$ be an upper canonical modification of f at $\hat{x}$ with canonical neighborhood $\Omega_2 \subset \Omega_1$, so that $f^{\#}|_{\Omega_2} \in C_P^2(\Omega_2)$ by Proposition 8.4 (i). We choose $\zeta(t, x) = \eta(t) + \{f^{\#}(x) - f(x)\}$ for $(t, x) \in Q$ with $\eta(t) = (t - \hat{t})^2$, so that (8.5) holds since the zero set of ζ is $\{\hat{t}\} \times M$ from Proposition 8.4(i). Then we observe that $\overline{s} = \hat{t}$ and

$$\psi(t, x) = g(t) + \eta(t) + f^{\#}(x) \quad \text{for } (t, x) \in Q_2 = I \times \Omega_2,$$

which belongs to $A_P(Q_2)$. Inequality (8.6) becomes

$$g'(s_k) + \eta'(s_k) + F(s_k, (f^{\#})'(y_k), \Lambda_W(f^{\#}, y_k)) \leqq 0. \tag{8.7}$$

Case (i) (f is not P-faceted at $\hat{x}$ in Ω). By Proposition 8.4(ii), we see $\overline{y} = \hat{x}$ and $(f^{\#})^{(n)}(\hat{x}) = f^{(n)}(\hat{x})$ for $n = 0, 1, 2$. Sending k to $+\infty$ in inequality (8.7), we conclude that (8.4) holds since we assumed continuity (F1) and ψ is continuously differentiable with respect to t, and is twice continuously differentiable with respect to x.

Case (ii) (f is P-faceted at $\hat{x}$ in Ω and $\chi(f, \hat{x}) \geqq 0$). Let I' be an open interval containing $\hat{t}$ with $\overline{I'} \subset I$. We see that $\{(t, x) \in \Omega;\ \zeta(t, x) = 0\} = \{\hat{t}\} \times R(f, \hat{x})$.

(A) Suppose that there exists a sequence $\{k_j\}_{j=1}^{\infty}$ such that $(s_{k_j}, y_{k_j}) \in \overline{I'} \times R(f, \hat{x})$, so that $f^{\#}$ is P-faceted at y_{k_j} in Ω_2 with slope 0. Owing to the definition of weighted curvature at the boundary of a faceted region, $\Lambda_W(f^{\#}, y_{k_j}) = \Lambda_W(f, \hat{x})$ by Proposition 8.4(i). So (8.7) becomes

$$g'(s_{k_j}) + \eta'(s_{k_j}) + F(s_{k_j}, 0, \Lambda_W(f^{\#}, \hat{x})) \leqq 0.$$

Sending j to $+\infty$, we have (8.4) by the continuity assumption (F1).

(B) Consider the negation of (A): For all $k > 0$, $(s_k, y_k) \in \{I \backslash \overline{I'}\} \times \{\Omega_2 \backslash R(f, \hat{x})\}$. We observe that $\overline{y} \in \partial R(f, \hat{x})$. Setting

$$\lambda(y) = W''((f^\#)'(y))(f^\#)''(y) \quad \text{for } y \in \Omega_2,$$

we have $\Lambda_W(f^\#, y_k) = \lambda(y_k)$ since $f^\#$ is not P-faceted at y_k in Ω_2. We see that $\lim_{k\to+\infty}(f^\#)'(y_k) = 0 = \lim_{k\to+\infty}(f^\#)''(y_k)$ since $f^\# \in C^2(\Omega_2)$, so that $\lim_{k\to+\infty} \lambda(y_k) = 0$ since W'' is bounded on every bounded set in $\boldsymbol{R}\backslash P$. Since inequality (8.7) becomes

$$g'(s_k) + \eta'(s_k) + F(s_k, (f^\#)'(y_k), \lambda(y_k)) \leqq 0,$$

we have

$$g'(\hat{t}) + F(\hat{t}, 0, 0) \leqq 0$$

by sending k to infinity. Since assumption $\chi(f, \hat{x}) \geqq 0$ yields $\Lambda_W(f, \hat{x}) \geqq 0$, we now get (8.4) by the degenerate ellipticity assumption (F2).

Case (iii) (f is P-faceted at $\hat{x}$ in Ω and $\chi(f, \hat{x}) = -1$). Fix $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is defined in 8.3. Since $(\overline{s}, \overline{y}) \in \{\hat{t}\} \times R(f, \hat{x})$ and $R(f, \hat{x}) \subset \operatorname{int} R(f^{\#,\varepsilon}, \hat{x})$, there exists $k_0 > 0$ such that $(s_k, y_k) \in I \times R(f^{\#,\varepsilon}, \hat{x})$ for all $k > k_0$. For $(s_k, y_k) \in I \times R(f^{\#,\varepsilon}, \hat{x})$, (8.7) becomes

$$g'(s_k) + \eta'(s_k) + F(s_k, 0, -\Delta/(L + 2\varepsilon)) \leqq 0,$$

where $\Delta = W'(+0) - W'(-0)$ and $L = L(f, \hat{x})$. Sending k to $+\infty$, we get

$$g'(\hat{t}) + F(\hat{t}, 0, -\Delta/(L + 2\varepsilon)) \leqq 0.$$

Since the last inequality holds for all $\varepsilon \in (0, \varepsilon_0)$, we get (8.4). $\square$

Lemma 8.1$'$ is proved as is Lemma 8.1.

Proof of Lemma 8.2. Step 1. Let $(\hat{t}, \hat{x}) \in Q$ and let $\varphi \in C(\Omega)$ be locally admissible at $(\hat{t}, \hat{x})$ in Q. Suppose that

$$\min_Q (v^* - \varphi) = (v^* - \varphi)(\hat{t}, \hat{x}).$$

Without loss of generality we may assume that $(v_* - \varphi)(\hat{t}, \hat{x}) = 0$ and we may also assume that $\varphi_x(\hat{t}, \hat{x}) = 0$ by Proposition 2.7.

Let ζ be a function on Q satisfying

$$\begin{aligned} &\zeta \in C(Q), \quad \zeta(\hat{t}, \hat{x}) = 0, \quad \zeta \geqq 0 \quad \text{in } Q, \\ &\psi \text{ is a locally admissible function at } (\hat{t}, \hat{x}) \text{ in } Q, \end{aligned} \tag{8.8}$$

where $\psi = \varphi - \zeta$ in Q. Function ζ is to be determined later. So we see that

$$\min_Q (v^* - \psi) = (v^* - \psi)(\hat{t}, \hat{x}),$$

$$\zeta \leqq v_* - \psi \quad \text{in } Q. \tag{8.9}$$

Suppose that by choosing ζ suitably there exists a rectangular neighborhood N_1 at $(\hat{t}, \hat{x})$ in Q satisfying

$$\psi_t(t,x) + F(t, \psi_x(t,x), \Lambda_W(\psi(t,\cdot),x)) < 0 \quad \text{for all } (t,x) \in N_1, \tag{8.10}$$

$$\psi|_{N_1} \in A_P(N_1), \tag{8.11}$$

$$N_0 \subset N_1 \quad \text{with } N_0 = \{(t,x) \in Q; \zeta(t,x) = 0\}. \tag{8.12}$$

Let N_2 be a rectangular (open) neighborhood at $(\hat{t}, \hat{x})$ satisfying $N_0 \subset N_2$ and $\overline{N_2} \subset N_1$. There exists $\sigma_1 > 0$ such that

$$\psi + \sigma_1 < h_* \quad \text{in } N_2. \tag{8.13}$$

In fact, from (8.9) and the definition of S, we obtain $\psi \leqq v_* \leqq h_*$ in Q. If there exists $(t_1, x_1) \in \overline{N_2}$ such that $\psi(t_1, x_1) = h_*(t_1, x_1)$, then the locally admissible function ψ is a test function of h at (t_1, x_1) in Q, which contradicts (8.10). So we have $\psi < h_*$ in N_2, or there exists $\sigma_1 > 0$ such that (8.13).

Since $\sigma_2 = \inf\{\zeta(x);\, x \in N_1 \backslash N_2\} > 0$ by the definition of N_2, we have

$$\psi + \sigma_2 \leqq v_* \quad \text{in } N_1 \backslash N_2, \tag{8.14}$$

which yields

$$\psi + \sigma \leqq h_* \quad \text{in } N_1 \tag{8.15}$$

with $\sigma = \min(\sigma_1, \sigma_2)$. By Propositions 2.7 and 2.8 we conclude that $\psi + \sigma$ is a subsolution of (E) in N_1.

We define $w(t,x)$ by

$$w(t,x) = \begin{cases} \max\{\psi(t,x) + \sigma, v(t,x)\}, & (t,x) \in N_2, \\ v(t,x), & (t,x) \in Q \backslash N_2. \end{cases}$$

Inequality (8.14) yields

$$w(t,x) = \max\{\psi(t,x) + \sigma, v(t,x)\} \quad \text{for } (t,x) \in N_1.$$

So w is a subsolution of (E) in N_1 by Lemma 8.1′.

To show that w is a subsolution of (E) in Q, suppose that $\psi_1 \in A_P(Q)$ satisfies

$$\max_Q (w - \psi_1) = (w - \psi_1)(t_0, x_0) = 0.$$

We may assume that

$$(t_0, x_0) \in N_2, \quad \psi_1(t_0, x_0) > v(t_0, x_0)$$

since otherwise ψ_1 is a test function of v at (t_0, x_0) so that

$$(\psi_1)_t(t_0, x_0) + F(t_0, (\psi_1)_x(t_0, x_0), \Lambda_W(\psi_1(t_0,\cdot), x_0)) \leqq 0. \tag{2.1′}$$

We may assume that $\psi_1(t_0, \cdot)$ is faceted at x_0 with slope $(\psi_1)_x(t_0, \cdot) \in P$ and that $R = R(\psi_1(t_0,\cdot), x_0)$ is not included in an interval J_1 with $N_1 = J_1 \times I_1$. Indeed, if not, (2.1′) holds by Proposition 6.19 since w is a subsolution of (E) in N_1. We may also assume that

$$\psi_1(t_0,x) > v(t_0,x) \quad \text{for } x \in R \cap J_1.$$

Indeed, if not, there is $x_1 \in R \cap J_1$ with $\psi_1(t_0,x_1) \leqq v(t_0,x_1)$. Since $v \leqq w \leqq \psi_1$ in Q, we now observe that

$$\max_Q (v - \psi_1) = (v - \psi_1)(t_0, x_1)$$

which yields (2.1′) with x_0 replaced by x_1. Since v is a subsolution of (E) in Q and since $x_1 \in R$ implies $(\psi_1)_x(t_0,x_0) = (\psi_1)_x(t_0,x_1)$, we now have (2.1′) without replacing x_0 by x_1. We may now assume that

$$\psi_1(t,x) > v(t,x) \quad \text{on } N_1,\ \ x \in R \cap J_1,$$

by taking I_1 smaller since v is upper-semicontinuous. We now modify $\psi_1(t_0,\cdot)$ in $R \cap (J_1 \backslash J_2)$ with $N_2 = J_2 \times I_2$ to get $\psi_2 \in A_P(N_1)$ satisfying

$$\begin{gathered}
\max_{N_1} (w - \psi_2) = (w - \psi_2)(t_0, x_0), \\
\operatorname{int} R(\psi_2(t_0,\cdot), x_0) \subset N_1, \\
\Lambda_W(\psi_2(t_0,\cdot), x_0) \leqq \Lambda_W(\psi_1(t_0,\cdot), x_0), \\
\psi_1 = \psi_2 \quad \text{in } N_2.
\end{gathered}$$

Since w is a subsolution in N_1 (even for the new choice of I_1 by Proposition 6.19) this yields (2.1′) with ψ_1 replaced by ψ_2. By (F2) we now have (2.1′). By (8.15) we now conclude that w is a subsolution of (E) in Q and that $w \in S$.

On the other hand, we have

$$0 = (v_* - \psi)(\hat{t}, \hat{x}) = \liminf_{d \downarrow 0} \{(v - \psi)(t,x); (t,x) \in N_2, |t - \hat{t}| < d, |x - \hat{x}| < d\},$$

which implies that there exists $(s,y) \in N_2$ such that $v(s,y) - \psi(s,y) < \sigma$. We now obtain $v(s,y) < w(s,y)$.

Step 2. We prove that there exists ζ satisfying (8.8) and N_1 satisfying (8.10)–(8.12). Since φ is locally admissible at $(\hat{t},\hat{x})$ in Q, there exists a rectangular neighborhood $Q_1 = I \times \Omega_1$ at $(\hat{t},\hat{x})$ in Q such that $\varphi|_{Q_1} \in A_P(Q_1)$. So there exists $f \in C_P^2(\Omega_1)$ and $g \in C^1(I)$ such that

$$\varphi(t,x) = f(x) + g(t) \quad \text{for } (t,x) \in Q_1.$$

Let $f_\# \in C(\Omega)$ be a lower canonical modification of f at $\hat{x}$ with canonical neighborhood $\Omega_2 \subset \Omega_1$. We chose $\zeta(t,x) = \eta(t) + \{f(x) - f_\#\}$ ($\geqq 0$) for $(t,x) \in Q$ with $\eta(t) = (t - \hat{t})^2$, so that

$$\psi(t,x) = g(t) - \eta(t) + f_\#(x) \quad \text{for } (t,x) \in N_3 = I \times \Omega_2.$$

Proposition 8.4(i) yields (8.8) and $N_0 = \{\hat{t}\} \times M$, where M is the same as (8.2).

Since v is not a supersolution of (E) and $v_* > -\infty$ in $[0,T) \times \overline{\Omega}$, we have

$$\psi_t(\hat{t},\hat{x}) + F(\hat{t},\psi_x(\hat{t},\hat{x}),\Lambda_W(\psi(\hat{t},\cdot),\hat{x})) < 0,$$

or

$$g'(\hat{t}) - \eta'(\hat{t}) + F(\hat{t},0,\Lambda_W(f_\#,\hat{x})) < 0$$

by Definition 6.7 of the local version of a supersolution and by Theorem 6.8. Clearly, there exists $\delta > 0$ such that

$$g'(\hat{t}) - \eta'(\hat{t}) + F(\hat{t},0,\Lambda_W(f_\#,\hat{x})) < -\delta.$$

For $(t,x) \in N_3$ we have

$$\begin{aligned} g'(t) &+ F(t,(f_\#)'(x),\Lambda_W(f_\#,x)) \\ &< \{g'(t) - g'(\hat{t}) + \eta'(\hat{t})\} \\ &\quad + \{F(t,(f_\#)'(x),\Lambda_W(f_\#,x)) - F(\hat{t},0,\Lambda_W(f,\hat{x}))\} \\ &\quad + \{F(\hat{t},0,\Lambda_W(f,\hat{x})) - F(\hat{t},0,\Lambda_W(f_\#,\hat{x}))\} - \delta. \end{aligned}$$

Let T_1, T_2 and T_3 denote the first, second and third term of the right-hand side of the last inequality. We shall show that there exists an open set N_1 such that $N_0 \subset N_1 \subset N_3$ and for all $(t,x) \in N_1$, $T_1 + T_2 + T_3 - \delta < 0$ holds. This now yields

$$g'(t) + F(t,(f_\#)'(x),\Lambda_W(f_\#,x)) < 0 \quad \text{for } (t,x) \in N_1,$$

which equals (8.10).

By Proposition 8.4(i), we see that $\Lambda_W(f_\#,\hat{x}) \leqq \Lambda_W(f,\hat{x})$, so that

$$T_3 \leqq 0 \tag{8.16}$$

holds by the degenerate ellipticity condition (F2). Since $g, \eta \in C^1(I)$, there exists $\rho_1 \in (0, \operatorname{dist}(\hat{t},\partial I))$ such that

$$T_1 = g'(t) - g'(\hat{t}) + \eta'(\hat{t}) < \tfrac{1}{2}\delta \quad \text{for all } t \in B(\hat{t},\rho_1), \tag{8.17}$$

where $B(\hat{t},\rho_1)$ denotes an open ball in $\boldsymbol{R}$ with center $\hat{t}$ and radius ρ_1. From the continuity condition (F1), there exists $\rho_2 > 0$ such that

$$|F(t,p,X) - F(\hat{t},0,0)| < \tfrac{1}{2}\delta \quad \text{for all } t,p \text{ and } X \in B(0,\rho_2). \tag{8.18}$$

From Proposition 8.4(iii), there exists an open interval Ω_3 such that

$$M \subset \Omega_3 \subset \Omega_2,$$

$$|(f_\#)'(x)| < \rho_2 \quad \text{for } x \in \Omega_3,$$

$$|\Lambda_W(f_\#,x) - \Lambda_W(f,\hat{x})| < \rho_2 \quad \text{for } x \in \Omega_3.$$

We chose $N_1 = B(\hat{t},\min(\rho_1,\rho_2)) \times \Omega_3$, so that (8.11) and (8.12) hold. Then we get $T_2 < \frac{1}{2}\delta$ for $(t,x) \in N_1$. We now have

$$T_1 + T_2 + T_3 - \delta < 0 \qquad \text{for } (t,x) \in N_1,$$

so that we conclude that (8.10) holds for $(t,x) \in N_1$. □

9. Existence Theorem for Periodic Initial Data

We give the proof of Existence Theorem 3.5 for periodic initial data. Throughout this section, let $T > 0, S \in (0,T)$ and $Q_S = (0,S) \times \boldsymbol{R}$. We often use the condition

(F1′) $\qquad F$ is continuous in $[0,S] \times \boldsymbol{R} \times \boldsymbol{R}$ with values $\boldsymbol{R}$.

The key tools to prove the Existence Theorem are Comparison Theorem 3.1 and the Perron-type Existence Theorem 3.4 together with the following lemma:

9.1. Lemma (Existence of Super- and Subsolutions). *Assume that conditions* (F1′) *and* (F2) *hold. Suppose that u_0 is bounded and uniformly continuous on $\boldsymbol{R}$. Then for each $S \in (0,T)$ there exists an upper-semicontinuous function $u^+ (= u^{+,S})$ and a lower-semicontinuous function $u^- (= u^{-,S})$ on $\bar{Q}_S$ such that u^+ and u^- are respectively a super- and subsolution of* (E) *in Q_S, and*

$$\begin{aligned} u^+(0,x) &= u_0(x) \quad \textit{for } x \in \boldsymbol{R}, \\ u^+(t,x) &\geqq u_0(x) \quad \textit{for } (t,x) \in \bar{Q}_S. \end{aligned} \tag{9.2}$$

$$\begin{aligned} u^-(0,x) &= u_0(x) \quad \textit{for } x \in \boldsymbol{R}, \\ u^-(t,x) &\leqq u_0(x) \quad \textit{for } (t,x) \in \bar{Q}_S. \end{aligned} \tag{9.2′}$$

9.2. Remark. If u_0 is periodic with period ϖ, the lemma holds with the extra property

$$u^+(t, x+\varpi) = u^+(t,x) \quad \text{for } (t,x) \in \bar{Q}_S. \tag{9.3}$$

or

$$u^-(t, x+\varpi) = u^-(t,x) \quad \text{for } (t,x) \in \bar{Q}_S. \tag{9.3′}$$

To prove Lemma 9.1, we extend the method developed in [CGG] and [IS] to C_P^2 functions. We carry out the proof in several steps.

9.3. Proposition. *Let M be a positive number. For any δ there exists $f_\delta (= f_\delta^M) \in C_P^2(\boldsymbol{R})$ such that*

$$f_\delta(0) = 0, \quad f_\delta, f_\delta'' \geqq 0 \ \textit{ in } \boldsymbol{R}, \quad f_\delta(x) \geqq M \ \textit{ for } x, |x| > \delta.$$

Proof. We set

$$V_0(x) = \begin{cases} (x+1)^4 & \text{for } x < -1, \\ 0 & \text{for } -1 \leqq x \leqq 1, \\ (x-1)^4 & \text{for } 1 < x. \end{cases}$$

By Lemma 6.12, there exists $V_1 \in C_P^2(\boldsymbol{R})$ such that

$$V_1(0) = 0 \quad \text{and} \quad V_1'' \geqq 0, \ V_1 \geqq V_0 \quad \text{in } \boldsymbol{R}.$$

Clearly there exists $k = k(\delta) \in (0, \frac{1}{2})$ with $kV_0(\delta/k) \geqq M$. Setting $f_\delta(x) = kV_1(x/k)$ for $x \in \boldsymbol{R}$, we see that $f_\delta \in C_P^2(\boldsymbol{R})$ since $f_\delta'(x) = V_1'(x/k)$ and V_1 is

P-faceted at $\hat{x}/k$ in $\boldsymbol{R}$ if and only if f_δ is P-faceted at $\hat{x}$ in $\boldsymbol{R}$. The other properties are easy to prove. $\square$

9.4. Lemma (Modification of C_P^2 Functions). *Let $\Omega_1 = (a_1, a_1'), \Omega_2 = (a_2, a_2')$ and Ω_3 be (nonempty) open intervals with $\overline{\Omega_3} \subset \Omega_2$ and $\overline{\Omega_2} \subset \Omega_1$. Suppose that $f \in C_P^2(\Omega_1)$ satisfies $f'' \geqq 0$ in Ω_2. Then there exists $V \in C_P^2(\Omega_1)$ such that*

$$V'' \geqq 0 \text{ on } \Omega_1, \quad V = f \text{ on } \Omega_3,$$

$$V'(x) = \begin{cases} q, & \text{for } x \in (a_1, a_2], \\ q', & \text{for } x \in [a_2', a_1') \end{cases}$$

with some q and $q' \notin P$. In particular, the number of faceted regions of V is finite.

Proof. We denote $\Omega_3 = (a_3, a_3')$. If there exists an open interval $I = (b_1, b_2)$ with

$$\overline{I} \subset (a_2, a_3), \ \ f'(x) \notin P \ \text{ for all } x \in I,$$

then we set $q = f'(b_1)$. Otherwise we have $q_1 = f'(x) \in P$ for all $x \in (a_2, a_3)$. In the latter case we choose an open interval I with $\overline{I} \subset (a_2, a_3)$ and choose $q \in (\sup\{q_2 \in P \cap \{-\infty\}; q_2 < q_1\}, q_1)$.

Likewise, if there exists an open interval $I' = (b_2', b_1')$ with

$$\overline{I'} \subset (a_3', a_2'), \ \ f'(x) \notin P \ \text{ for all } x \in I',$$

then we set $q' = f'(b_1')$. Otherwise we have $q_1' = f'(x) \in P$ for all $x \in (a_3', a_2')$. In the latter case we choose an open interval I' with $\overline{I'} \subset (a_3', a_2')$ and choose $q' \in (q_1', \inf\{q_2' \in P \cup \{+\infty\}; q_2' > q_1'\})$.

Let $h_1 \in C^1(\Omega_1)$ with

$$h_1(x) = \begin{cases} q & \text{for } x \in (a_1, b_2), \\ q' & \text{for } x \in (b_2', a_1'). \end{cases}$$

We connect f' and h_1 in the following way. Let $\rho_1, \rho_2 \in C^1(\Omega_1)$ satisfy

$$\rho_1 + \rho_2 = 1 \text{ on } \Omega_1, \quad 0 \leqq \rho_1, \rho_2 \leqq 1 \text{ on } \Omega_1,$$

$$\rho_1(x) = \begin{cases} 1 & \text{for } x \in \overline{I_2}, \\ 0 & \text{for } x \in \Omega_1 \backslash I_1, \end{cases} \qquad \rho_1'(x) = \begin{cases} \geqq 0 & \text{for } x \in (b_1, b_2), \\ \leqq 0 & \text{for } x \in (b_2', b_1'), \end{cases}$$

$$\rho_2(x) = \begin{cases} 0 & \text{for } x \in \overline{I_2}, \\ 1 & \text{for } x \in \Omega_1 \backslash I_1, \end{cases} \qquad \rho_2'(x) = \begin{cases} \leqq 0 & \text{for } x \in (b_1, b_2), \\ \geqq 0 & \text{for } x \in (b_2', b_1'), \end{cases}$$

where we denote $I_1 = (b_1, b_1')$ and $I_2 = (b_2, b_2')$, so that $\overline{I_2} \subset I_1$.

We set

$$h_2 = \rho_1 f' + \rho_2 h_1 \quad \text{in } \Omega_1$$

to get

$$h_2' = \rho_2'(h_1 - f') + \rho_1 f'' \geqq 0 \quad \text{in } \Omega_1,$$

since

$$h_1 \leqq f' \text{ in } (b_1, b_2), \quad h_1 \geqq f' \text{ in } (b_2', b_1').$$

Thus we get the desired function $V \in C_P^2(\Omega_1)$ by setting

$$V(x) = f(a) + \int_a^x h_2(y)\, dy \quad \text{for } x \in \Omega_1$$

with some $a \in \Omega_3$. $\square$

9.5. Lemma. *Let M be a positive number.*
(i) *For any $\delta \in (0, \frac{1}{2})$ there exists V_δ $(= V_\delta^M) \in C_P^2(\boldsymbol{R})$ such that*

$$V_\delta(0) = 0, \quad V_\delta, V_\delta'' \geqq 0 \textit{ in } \boldsymbol{R}, \quad V_\delta \geqq M \quad \textit{for } x, |x| > \delta,$$

$$V_\delta'(x) = \begin{cases} q_\delta, & \textit{for } 1 \leqq x, \\ q_\delta', & \textit{for } x \leqq -1 \end{cases}$$

with some q_δ $(= q_\delta^M)$ and q_δ' $(= q_\delta'^M) \notin P$.
(ii) *Assume that condition* (F1$'$) *holds. For each $S \in (0, T)$ there exists a* (*large*) *B_δ $(= B_\delta^{M,S}) \geqq 0$ such that $V_\delta^+ \in A_P(Q_S)$ is a supersolution of* (E) *in $Q_S = (0, S) \times \boldsymbol{R}$ of the form*

$$V_\delta^+(t, x)(= V_\delta^{+,M,S}|_{Q_S}) = Bt + V_\delta(x) \quad \textit{for } (t, x) \in \overline{Q}_S \textit{ and } B \geqq B_\delta.$$

Here the dependence of M and S on V_δ^+ and B_δ is suppressed.

Proof. It is clear that Lemmas 9.3 and 9.4 with $\Omega_1 = \boldsymbol{R}, \Omega_2 = (-1, 1)$ and $\Omega_3 = (-\frac{1}{2}, \frac{1}{2})$ yield (i). Now we give the proof of (ii). We see that $V_\delta^{+,M,S}|_{Q_S} \in A_P(Q_S)$ by the definition.

Since the number of faceted regions of V_δ is finite, we have

$$c_1 = \sup\{|\Lambda_W(V_\delta, x)|;\ x \in \boldsymbol{R}, V_\delta'(x) \in P\} < \infty.$$

We also have

$$c_2 = \sup\{|\Lambda_W(V_\delta, x)|;\ x \in \boldsymbol{R}, V_\delta'(x) \notin P\} < \infty,$$

since

$$\sup\{|V_\delta''(x)|;\ |x| \leqq 1\} < \infty,$$
$$\sup\{|W''(p)|;\ q_\delta \leqq p \leqq q_\delta',\ p \notin P\} < \infty$$

by the assumption of W. Thus we observe that

$$\sup\{|\Lambda_W(V_\delta, x)|;\ x \in \boldsymbol{R}\} < c_3 < \infty$$

with $c_3 = \max(c_1, c_2)$.

Since condition (F1′) yields

$$c_4 = \inf\{F(t,p,X);\quad t \in [0,S],\ p \in [q_\delta, q'_\delta],\ |X| \leqq c_3\} > -\infty,$$

we have

$$F(t, V'_\delta(x), \Lambda_W(V_\delta, x)) \geqq c_4 \quad \text{for all } (t,x) \in \overline{Q}_S.$$

We take B_δ so that $B_\delta \geqq \max(-c_4, 0)$. Function V^+_δ is a supersolution of (E) in Q_S. In fact, if $((\hat{t},\hat{x}),\psi) \in Q_S \times A_P(Q_S)$ satisfies

$$\min_{Q_S}(V^+_\delta - \psi) = (V^+_\delta - \psi)(\hat{t},\hat{x}),$$

then

$$\begin{aligned}\psi_t(\hat{t},\hat{x}) + F(\hat{t}, \psi_x(\hat{t},\hat{x})), \Lambda_W(\psi(\hat{t},\cdot),\hat{x}) &= B + F(\hat{t}, V^+_\delta(\hat{x}), \Lambda_W(V^+_\delta, \hat{x})) \\ &\geqq B - B_\delta \geqq 0\end{aligned}$$

since $V_\delta \in C^2_P(\boldsymbol{R})$. It is clear that $(V_\delta|_{Q_S})_* \geqq 0 \geqq -\infty$ in $[0,S) \times \boldsymbol{R}$. □

9.6. Lemma. *Suppose that u_0 is bounded and uniformly continuous in $\boldsymbol{R}$, so that for each $\varepsilon \in (0,1)$ there exists $\delta = \delta(\varepsilon) \in (0, \frac{1}{2})$ satisfying*

$$|u_0(x) - u_0(\xi)| < \varepsilon \quad \textit{for } |x - \xi| < \delta. \tag{9.4}$$

Let $V_{\delta(\varepsilon)} \in C^2_P(\boldsymbol{R})$ be as in Lemma 9.5(i) with $M = \max_{x\in\boldsymbol{R}} u_0(x) - \min_{x\in\boldsymbol{R}} u_0(x)$. Then we have

$$u_0(x) = \inf\{V_{\delta(\varepsilon)}(x-\xi) + u_0(\xi) + \varepsilon;\quad \varepsilon \in (0,1),\ \xi \in \boldsymbol{R}\}.$$

Proof. Since (9.4) implies

$$u_0(x) < u_0(\xi) + \varepsilon \quad \text{for } |x - \xi| \leqq \delta(\varepsilon)$$

and since the definition of M implies

$$u_0(x) < u_0(\xi) + M \leqq u_0(\xi) + V_{\delta(\varepsilon)}(x-\xi) \quad \text{for } |x-\xi| > \delta(\varepsilon),$$

we have

$$u_0(x) \leqq V_{\delta(\varepsilon)}(x-\xi) + u_0(\xi) + \varepsilon \quad \text{for } x,\xi \in \boldsymbol{R},\ \varepsilon \in (0,1).$$

For each $x \in \boldsymbol{R}$ and $\varepsilon' > 0$, we have

$$V_{\delta(\varepsilon)}(x-\xi) + u_0(\xi) + \varepsilon < \varepsilon' + u_0(x)$$

with $\xi = x$ and $\varepsilon = \frac{1}{2}\varepsilon'$, which yields the results. □

Proof of Lemma 9.1. Firstly, for each $\varepsilon \in (0,1)$ let $V_{\delta(\varepsilon)} \in C^2_P(\boldsymbol{R})$ be the same as in Lemma 9.6. Secondly, for each $\varepsilon \in (0,1)$ and $\xi \in \boldsymbol{R}$ we set

$$u^{+,\varepsilon}(t,x;\xi) = V^+_{\delta(\varepsilon)}(t, x-\xi) + u_0(\xi) + \varepsilon \quad \text{for } (t,x) \in \overline{Q}_S$$

with $B = B_{\delta(\varepsilon)}$. Then $u^{+,\varepsilon}(t,x;\xi)$ belongs to $A_P(Q_S)$ and is a supersolution of (E) in Q_S by Lemma 9.5(ii). Lastly, we take

$$u^+(t,x) = \inf\{u^{+,\varepsilon}(t,x,\xi); \varepsilon \in (0,1), \xi \in \boldsymbol{R}\} \quad \text{for } (t,x) \in \overline{Q}_S;$$

then u^+ is upper-semicontinuous in $\overline{Q}_S$ and

$$(u^+|_{Q_S})_* \geqq \min_{\xi \in \boldsymbol{R}} u_0(\xi) > -\infty \quad \text{for } [0,S) \times \boldsymbol{R}.$$

So, Lemma 8.1′ yields that u^+ is a supersolution of (E) in Q_S. Now we have (9.2). In fact, Lemma 9.6 implies that

$$u^+(0,x) = u_0(x) \quad \text{for } x \in \boldsymbol{R},$$

$$u^+(t,x) \geqq u_0(x) \quad \text{for } (t,x) \in \overline{Q}_S,$$

since $B_{\delta(\varepsilon)} \geqq 0$ and

$$V^+_{\delta(\varepsilon)}(t, x-\xi) \geqq V_{\delta(\varepsilon)}(x-\xi) \quad \text{for } (t,x) \in \overline{Q}_S,\ \xi \in \boldsymbol{R}.$$

We can likewise get the lower-semicontinuous function u^- satisfying the desired property. □

Proof of Remark 9.2. If u_0 is periodic with period ϖ, the function u^+ constructed in the proof of Lemma 9.1 satisfies (9.3), since

$$\begin{aligned} u^{+,\varepsilon}(t, x+\varpi; \xi) &= V^+_{\delta(\varepsilon)}(t, x+\varpi-\xi) + u_0(\xi) + \varepsilon \\ &= V^+_{\delta(\varepsilon)}(t, x-(\xi-\varpi)) + u_0(\xi-\varpi) + \varepsilon \\ &= u^{+,\varepsilon}(t,x,\xi-\varpi). \quad \square \end{aligned}$$

Proof of Theorem 3.5. Step 1 (Existence on Q_S). Since (F1) implies (F1′) for each $S \in (0,T)$, Lemma 9.1 is applicable. For each $S \in (0,T)$, let u^+ and u^- be an upper-, and a lower-semicontinuous function in Q_S obtained in Lemma 9.1. By Theorem 3.4, there exists a generalized solution $\tilde{u}$ of (E) in Q_S such that

$$u^- \leqq \tilde{u} \leqq u^+ \quad \text{in } Q_S,$$

$$\tilde{u}(t, x+\varpi) = \tilde{u}(t,x) \quad \text{for } (t,x) \in Q_S.$$

Since u^+ and u^- are respectively upper- and lower-semicontinuous in $\overline{Q}_S$, we have

$$-\infty < u^- \leqq \tilde{u}_* \leqq \tilde{u}^* \leqq u^+ < \infty \quad \text{in } \overline{Q}_S.$$

Since $u^-(0,x) = u^+(0,x) = u_0(x)$ in $\boldsymbol{R}$, we have

$$\tilde{u}_*(0,x) = \tilde{u}^*(0,x) = u_0(x) \quad \text{in } \boldsymbol{R}.$$

Theorem 3.2 yields

$$\tilde{u}^* \leqq \tilde{u}_* \quad \text{in } Q_S, \quad \text{that is, in } \overline{Q}_S,$$

which implies that $\tilde{u}^*$ is continuous in $\overline{Q}_S$. We set $u^S = \tilde{u}^*$ in $[0,S)$. Then u^S is a generalized solution of (E) in Q_S, and $u^S \in C([0,S) \times \boldsymbol{R})$ satisfies

$$u^S(0,x) = u_0(x) \qquad \text{for } x \in \boldsymbol{R},$$
$$u^S(t, x+\varpi) = u^S(t,x) \qquad \text{for } (t,x) \in [0,S) \times \boldsymbol{R}. \tag{9.5}$$

Step 2 (Uniqueness on Q_S). Let $v^S \in C([0,S)\times\boldsymbol{R})$ satisfy (9.5) with u^S replaced by v^S, such that v^S is a generalized solution of (E) in Q_S. Theorem 3.2 yields that $u^S = v^S$ in $[0,S)\times\boldsymbol{R}$, which implies the uniqueness of u^S in $[0,S)\times\boldsymbol{R}$.

Step 3 (Unique Existence on $(0,T)\times\boldsymbol{R}$). If $0 < S < S' < T$, Proposition 6.19 implies that $u^{S'}|_{Q_S}$ is a generalized solution of (E) in Q_S. By Step 2 for $t \in (0,T)$, it is possible to define u from $\{u^S; 0 < S < T\}$ uniquely by

$$u(t,x) = u^S(t,x) \quad \text{with } S \in (t,T).$$

From Proposition 6.20 we see that u is a generalized solution of (E) in $(0,T)\times\boldsymbol{R}$. It is easy to see that other properties hold. □

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References

[AT] F. ALMGREN & J. TAYLOR, *Flat flow is motion by crystalline curvature for curves with crystalline energies,* J. Diff. Geom. **42** (1995) 1–22.

[ATW] F. ALMGREN, J. TAYLOR & L. WANG, *Curvature driven flows: A variational approach,* SIAM J. Control Optim. **31** (1993) 387–437.

[AG1] S. B. ANGENENT & M. E. GURTIN, *Multiphase thermomechanics with interfacial structure 2. Evolution of an isothermal interface,* Arch. Rational Mech. Anal. **108** (1989) 323–391.

[AG2] S. B. ANGENENT & M. E. GURTIN, *Anisotropic motion of a phase interface,* J. reine angew. Math. **446** (1994) 1–47.

[CGG] Y.-G. CHEN, Y. GIGA & S. GOTO, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,* J. Diff. Geom. **33** (1991) 749–786.

[Ch] A. A. CHERNOV, *Modern Crystallography III,* Crystal Growth, Solid-State Sciences **36**, Springer 1984 (especially §3.3).

[CEL] M. G. CRANDALL, L. C. EVANS & P.-L. LIONS, *Some properties of viscosity solutions of Hamilton-Jacobi equations,* Trans. Amer. Math. Soc. **282** (1984) 487–502.

[CIL] M. G. CRANDALL, H. ISHII & P.-L. LIONS, *User's guide to viscosity solutions of second order partial differential equations,* Bull. Amer. Math. Soc. **27** (1992) 1–67.

[EGS] C. M. ELLIOTT, A. R. GARDINER & R. SCHÄTZLE, *Crystalline curvature flow of a graph in a variational setting,* Univ. of Sussex at Brighton, Centre for mathematical analysis and its applications, Research report No. 95/06.

[ES] L. C. Evans & J. Spruck, *Motion of level sets by mean curvature. I.* J. Diff. Geom. **33** (1991) 635–681.

[FK] I. Fonseca & M. Katsulakis, *Γ-convergence, minimizing movements and generalized mean curvature evolution,* Diff. Int. Eq. **8** (1995) 1619–1656.

[FG] T. Fukui & Y. Giga, *Motion of a graph by nonsmooth weighted curvature,* World Congress of Nonlinear Analysts '92 (ed. V. Lakshmikantham), de Gruyter, Berlin, vol. I, 1996, pp. 47–56.

[GMHG1] M.-H. Giga & Y. Giga, *Geometric evolution by nonsmooth interfacial energy,* Proc. of Banach Center Minisemester "Nonlinear Analysis Applications" (eds. N. Kenmochi et al.), Gakuto 1995, pp. 125–140.

[GMHG2] M.-H. Giga & Y. Giga, *Consistency in evolutions by crystalline curvature,* Proc. Free Boundary Problems '95, Theory and Applications (ed. M. Niezgódka and P. Strzelecki), Zakopane, Poland 1995, Longman 1996, pp. 186–202.

[GMHG3] M.-H. Giga & Y. Giga, *A subdifferential interpretation of crystalline motion under nonuniform driving force,* Hokkaido University Preprint Series in Mathematics #344, 1996.

[GMHG4] M.-H. Giga & Y. Giga, *Stability for evolving graphs by nonlocal weighted curvature,* Hokkaido University Preprint Series in Mathematics #372, 1997.

[GMHG5] M.-H. Giga & Y. Giga, *Remarks on convergence of evolving graphs by nonlocal curvature,* Proc. Third European Conference on Elliptic and Parabolic Problems (eds. H. Amann et al.), Pont-à-Mousson, France 1997 (to appear).

[G] Y. Giga, *Motion of a graph by convexified energy,* Hokkaido Math. J. **23** (1994) 185–212.

[GGo] Y. Giga & S. Goto, *Motion of hypersurfaces and geometric equations,* J. Math. Soc. Japan **44** (1992) 99–111.

[GGIS] Y. Giga, S. Goto, H. Ishii & M.-H. Sato, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains,* Indiana Univ. Math. J. **40** (1991) 443–470.

[GGu] Y. Giga & M. E. Gurtin, *A comparison theorem for crystalline evolution in the plane,* Quart. J. Appl. Math. **54** (1996) 727–737.

[GGuM] Y. Giga, M. E. Gurtin & J. Matias, *On the dynamics of crystalline motions,* Japan J. Indust. Appl. Math. **15** (1998) 1–44.

[Gir] P. M. Girão, *Convergence of a crystalline algorithm for the motion of a simple closed convex by weighted curvature,* SIAM J. Numer. Anal. **32** (1995) 886–899.

[GirK1] P. M. Girão & R. V. Kohn, *Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature,* Numer. Math. **67** (1994) 41–70.

[GirK2] P. M. Girão & R. V. Kohn, *The crystalline algorithm for computing motion by curvature, Variational Methods for Discontinuous Structure,* (eds. R. Serapioni & F. Tomarelli), Birkhäuser, 1996, pp. 7–18.

[Go] S. Goto, *Generalized motion of hypersurfaces whose growth speed depends superlinearly on the curvature tensor,* Diff. Int. Eq. **7** (1994) 323–343.

[Gu] M. E. Gurtin, *Thermomechanics of Evolving Phase Boundaries in the Plane,* Oxford, Clarendon Press, 1993.

[GSS] M. E. Gurtin, H. M. Soner & P. E. Souganidis, *Anisotropic motion of an interface relaxed by formulation of infinitesimal wrinkles,* J. Diff. Eq. **119** (1995) 54–108.

[I] H. Ishii *Perron's method for Hamilton-Jacobi equations.* Duke Math. J. **55** (1987) 369–384.

[IS] H. Ishii & P. Souganidis, *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor,* Tohoku Math. J. **47** (1995) 227–250.

[IR] H. ISHII & M. RAMASWAMY, *Uniqueness results for a class of Hamilton-Jacobi equations with singular coefficients,* Comm. Partial Diff. Eq. **20** (1995) 2187–2213.

[KK] B. KAWOHL & N. KUTEV, *Global behaviour of solutions to a parabolic mean curvature equation,* Diff. Int. Eq. **8** (1995) 1923–1946.

[Ko] Y. KŌMURA, *Nonlinear semi-groups in Hilbert space,* J. Math. Soc. Japan **19** (1967) 493–507.

[LL] J. M. LASRY & P.-L. LIONS, *A remark on regularization in Hilbert spaces,* Israel J. Math. **55** (1986) 257–266.

[LSU] O. A. LADYZHENSKAYA, V. SOLONNIKOV & N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type,* Amer. Math. Soc., 1968.

[LS] S. LUCKHAUS & T. STURZENHECKER, *Implicit time discretization for the mean curvature flow equations,* Calculus of Variations and Partial Diff. Eq. **3** (1995) 253–271.

[MBK] H. MÜLLER-KRUMBHAAR, T. W. BURKHARDT & D. M. KROLL, *A generalized kinetic equation for crystal growth,* J. Crystal Growth **38** (1977) 13–22.

[OKS] M. OHNUMA & K. SATO, *Singular degenerate parabolic equations with applications to the p-Laplace diffusion equation,* Comm. Partial Diff. Eq. **22** (1997) 381–411.

[OhS] M. OHNUMA & M.-H. SATO, *Singular degenerate parabolic equations with applications to geometric evolutions,* Diff. Int. Eq. **6** (1993) 1265–1280.

[R] A. R. ROOSEN, *Crystalline curvature and flat flow in a diffusion field,* preprint 1994.

[Ry] P. RYBKA, *A crystalline motion: uniqueness and geometric properties,* SIAM J. Appl. Math. **57** (1997) 53–72.

[St] A. STANCU, *Uniqueness of self-similar solutions for a crystalline flow,* Indiana Univ. Math. J. **45** (1996) 1157–1174.

[T1] J. TAYLOR, *Constructions and conjectures in crystalline nondifferential geometry,* In: Differential Geometry (eds. B. LAWSON & K. TANENBLAT), Proceedings of the Conference on Differential Geometry, Rio de Janeiro, Pitman Monographs in Pure and Applied Math. **52** (1991) 321–336, Pitman, London.

[T2] J. TAYLOR, *Mean curvature and weighted mean curvature,* Acta Metall. **40** (1992) 1475–1485.

[T3] J. TAYLOR, *Motion of curves by crystalline curvature, including triple junctions and boundary points,* In: Differential Geometry: Partial Differential Equations on Manifolds (eds. R. GREENE & S.T. YAU), Proc. Symp. Pure Math. **54** (1993), part I, pp. 417–438, Amer. Math. Society, Providence.

[TCH] J. TAYLOR, J. CAHN & C. HANDWERKER, *Geometric models of crystal growth,* Acta Metall. **40** (1992) 1443–1474.

[UY] T. USHIJIMA & S. YAZAKI, Convergence of a crystalline algorithm for the motion of a closed curve by a power of curvature $v = \kappa^{\alpha}$, preprint 1997.

[Y] K. YOSIDA, *Functional Analysis,* 6th edition, Springer, Berlin Heidelberg New York 1980.

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